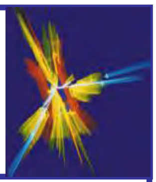


Relativistic kinematics

A 5 ECTS credit (3 sw) course autumn 2005

http://www.physics.helsinki.fi/~www_sefo/kinematics/

given by

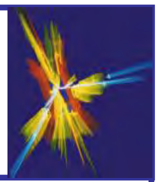
Kenneth Österberg,
Department of Physical Sciences,
University of Helsinki
room B323; reception thu 10-12

lectures: tue 14-16 in D116

exercise session: thu 9-10 in D116

lectures: weeks 36-50

exercises start week 38



Course literature/content/grading

Literature:

- **Lecture notes** – main source due to lack of good book (copies available on homepage & in folder on 2nd floor).
- E. Byckling & K. Kajantie: Particle kinematics (John Wiley & Sons 1973) – a rather theoretical & outdated text book, useful for side reading & solving exercises.
- PDG's kinematics review (link at course homepage).
- W. von Schlippe: Lectures on relativistic kinematics, St. Petersburg State University (see course homepage).

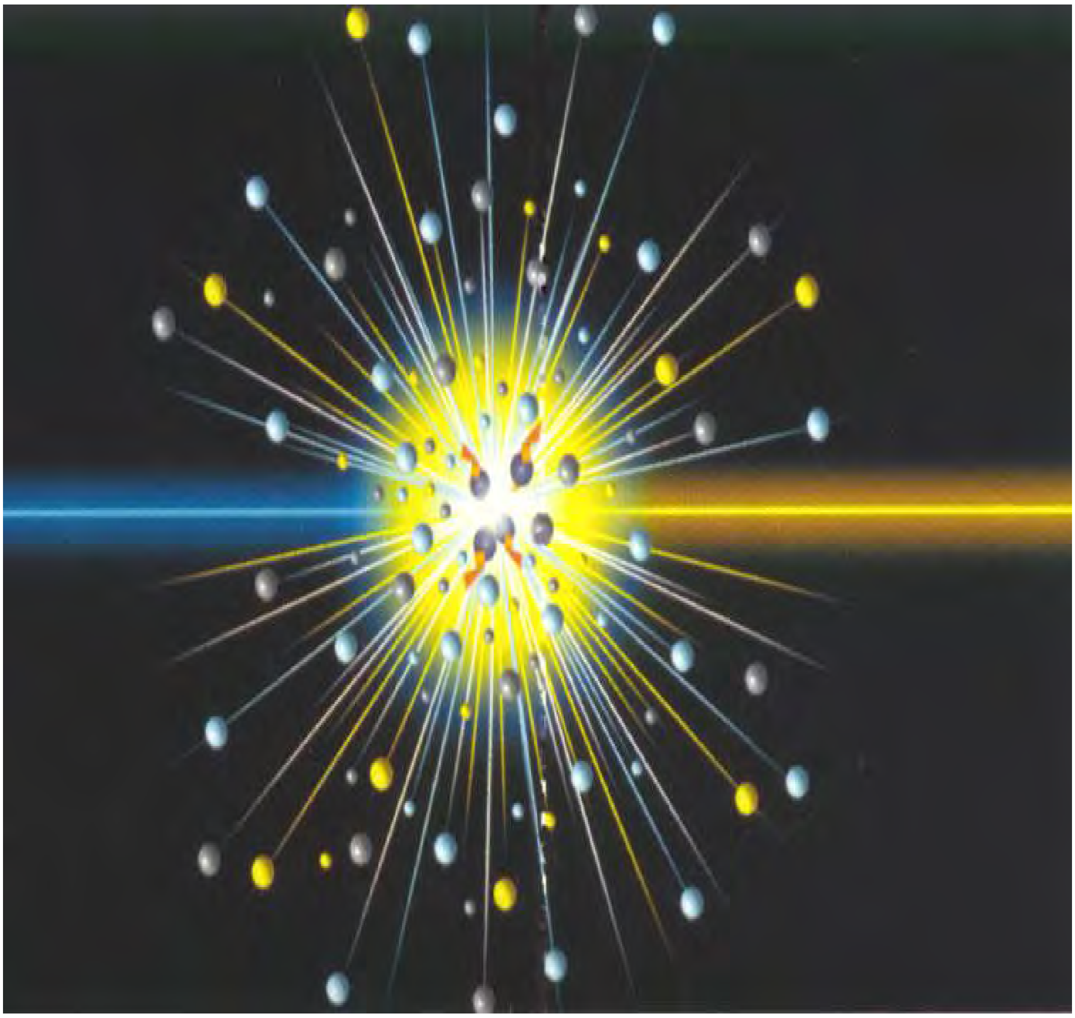
Content

- Special relativity: Lorentz transformations & invariants, reference frames, units and definitions.
- Phase space: phase space integral, total & differential cross section and Jacobian determinants.
- 2-particle final states: 2-particle decay & scattering, the Mandelstam variables and physically allowed regions.
- 3-particle final states: 3-particle decay & scattering.
- Inclusive reactions & multiparticle production
- Monte Carlo methods
- Monte Carlo event generators

Grading

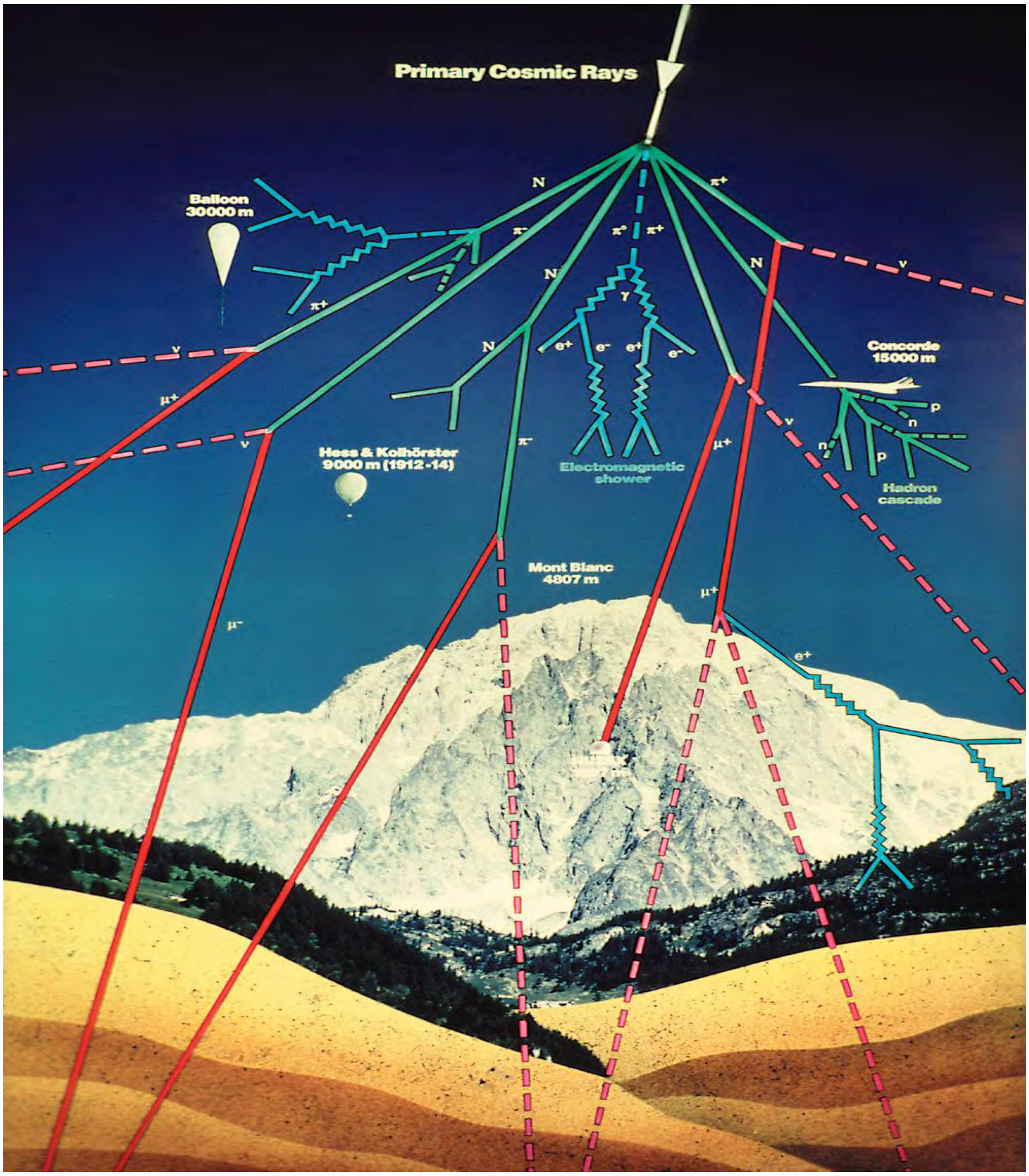
- Exercises (~ 8–9 exercise papers) – 30 % weight, Final exam at the end of the course – 70 % weight.
- Exercises given on Thursdays, to be returned by next Wednesday afternoon (1. exercise will be given next week)

Relativistic kinematics applied in high energy physics collisions at particle accelerators

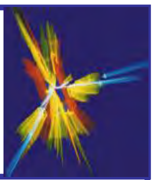


(an artist view of electron and positron interactions at LEP)

Or in describing cosmic rays



Cosmic rays: showers upto 10^{20} eV seen ⇒
$E^{CM} \approx$ 400 TeV (if target to be assumed proton)



The fundamental of special relativity is the fact that the velocity of light, c (= 2.99792 ... $*$ 10^8 m/s in vacuum), is the same in all inertial frames $\Rightarrow$ all measurements involving distances is influenced since light is supposed to be the fastest means of communication. The influence can be expressed by Lorentz transformations.

Take two reference frames S and S' moving with a constant velocity with respect to each other. A world point ("event") in space-time is defined by its coordinates x, y, z, t and x', y', z', t' in the respective frames. Assume a light signal is sent out from one world point ("P_1") and received in another world point ("P_2") then the distance $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

but also $d = c(t_2 - t_1) \Rightarrow (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 = c^2(t_2-t_1)^2$

same could be written in frame S' with primed coordinates and with the same constant c.

introduce $\tau = ict$ and go over to infinitesimal distances:

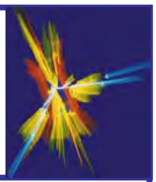
$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = -(dx^2 + dy^2 + dz^2 + d\tau^2)$$

From above conclude $ds = 0 \Rightarrow ds' = 0$

$$\int_{P_1}^{P_2} ds \text{ (in S frame)} = \int_{P'_1}^{P'_2} ds' \text{ (in S' frame)}$$ **invariant**

ds is called the invariant line element

Spacelike and timelike distances



Now given two events P_1 and P_2 with a certain distance in frame S, is there a frame S' in which these 2 events appear at the same time ($\Delta\tau' = 0$)?

$$\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta\tau^2) = -(\Delta x'^2 + \Delta y'^2 + \Delta z'^2) \le 0$$

A frame in which two events happen at the same **time** can only be found if and only if:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \le 0$$ **spacelike distance**

Can two events P_1 and P_2 appear to happen in the same place is some frame S' ($\Delta x'^2 + \Delta y'^2 + \Delta z'^2 = 0$)?

$$\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta\tau^2) = -\Delta\tau'^2 = c^2\Delta t'^2 \ge 0$$

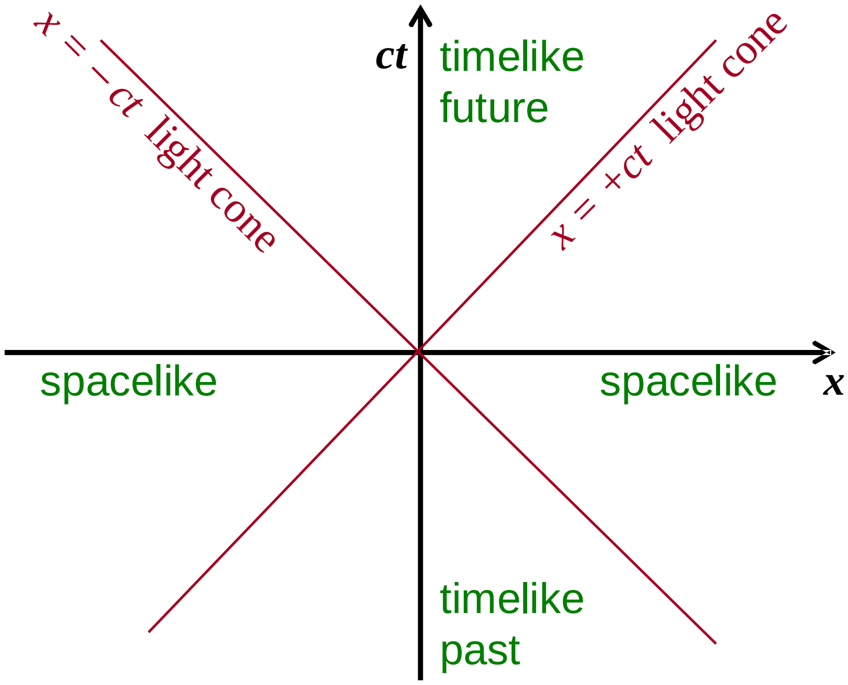
A frame in which two events happen in the same **place** can only be found if and only if:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \ge 0$$ **timelike distance**

Since Δs^2 is invariant hence either $\Delta s^2 \le 0$ in all Lorentz frames or $\Delta s^2 \ge 0$ in all Lorentz frames. There is also the case $\Delta s^2 = 0$ which applies to the distance between two events connected by a light signal (see previous page).

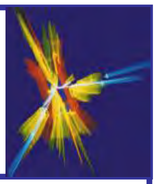
Causality and the light cone

Consider all possible events w. r. t. a given one, put into origo ($x = y = z = t = 0$) and draw only (x, t). The distance from origo is given by the invariant: $s^2 = c^2t^2 - x^2 - y^2 - z^2$



- $s^2 = 0$: connects events that can be reached by a light signal from origo (”lightlike”). Defines the ”light cone”.
- $s^2 > 0$: timelike events; if $s > 0$, the event is in the forward light cone (in the absolute future); if $s < 0$, the event is in the backward light cone (in the absolute past). Since s^2 is invariant, only events in the backward cone can have an influence on origo and origo can only have an influence on events in the forward cone.
- $s^2 < 0$: spacelike events; no interaction with origo.

causality expressed simply; very relevant for cosmology



invariance: $\Delta s^2 = -(\Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta\tau^2) = -(\Delta x'^2 + \Delta y'^2 + \Delta z'^2 + \Delta\tau'^2)$

If translations excluded, only transformations leaving Δs^2 invariant are rotations connected with τ; a rotation α in the z–τ plane is singled out by the figure below

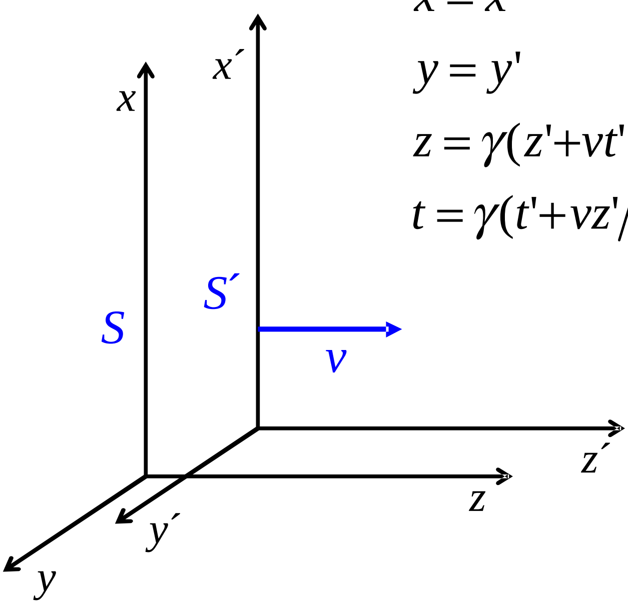
$$\left.\begin{aligned} z &= z'\cos\alpha - \tau'\sin\alpha \\ \tau &= z'\sin\alpha + \tau'\cos\alpha \end{aligned}\right\} \quad \text{trivially} \quad x = x' \text{ and } y = y'$$

determine α by being in frame S and observing $z´$=0 of $S´$:

$$\left.\begin{aligned} z &= -\tau'\sin\alpha \\ \tau &= \tau'\cos\alpha \end{aligned}\right\} \quad -\frac{z}{\tau} = -\frac{v}{ic} = \tan\alpha = i\frac{v}{c} \qquad \Rightarrow$$

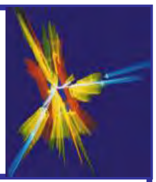
$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1-v^2/c^2}} \equiv \gamma \qquad \sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{iv/c}{\sqrt{1-v^2/c^2}} = i\gamma\frac{v}{c}$$

Lorentz transformations:



$$\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma(z' + vt') \\ t &= \gamma(t' + vz'/c^2) \end{aligned} \qquad \begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma(t - vz/c^2) \end{aligned}$$

$S´$ to S Lorentz transformations obtained by change of v to $-v$



The equations of previous page form a special class of Lorentz transformations but that is all that is needed here. The most general Lorentz transformation equations have the simplest form in four-vector space $x = (x^0, x^1, x^2, x^3) = (x^0, \vec{x}) = (ct, x, y, z)$. For any four-vector the general Lorentz transformation is given as:

$$a' = \mathbf{L}a \quad a'^{\mu} = \sum_{0}^{3} L^{\mu}_{\nu} a^{\nu}$$, where $\mathbf{L}$ is a real matrix

the metric tensor: $$\mathbf{g} = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{g}^{-1}$$

the scalar product: $$a \cdot b = \sum_{\mu} a_{\mu} b^{\mu} = \sum_{\mu,\nu} g_{\mu\nu} a^{\mu} b^{\nu} = a^0 b^0 - \bar{a} \cdot \bar{b}$$

A Lorentz transformation is a linear transformation that leaves the scalar product $a \cdot b$ invariant ($\Rightarrow$ $\mathbf{L}$ has to satisfy $\mathbf{g}\mathbf{L}^{-1}\mathbf{g} = \mathbf{L}^{T}$). Can be expressed as a boost (see previous page) followed by a 3-dimensional rotation. In addition, Lorentz transformations of the course satisfy following conditions:

$$\det \mathbf{L} = +1, \quad \text{i.e. spatial reflections excluded}$$
$$L^0_0 \geq 1, \quad \text{sign of 0-component of timelike vector invariant}$$

The specific Lorentz transformation solution of previous page would give:

$$a'^0 = \gamma(a^0 - va^3/c)$$
$$a'^1 = a^1$$
$$a'^2 = a^2$$
$$a'^3 = \gamma(a^3 - va^0/c)$$

Rapidity

The Lorentz transformations form a group, i.e. the product of two Lorentz transformations is again a Lorentz transformation. If one performs consecutively two Lorentz transformations with parameters v_1 and v_2:

$$v_3 = (v_1 + v_2)/(1 + v_1 v_2 / c^2) \quad \gamma_3 = \gamma_1 \gamma_2 (1 + v_1 v_2 / c^2)$$

More clearly visible in terms of a new parameter ξ rapidity

$$v/c = \tanh \xi \quad \gamma = \cosh \xi \quad \gamma v/c = \sinh \xi$$

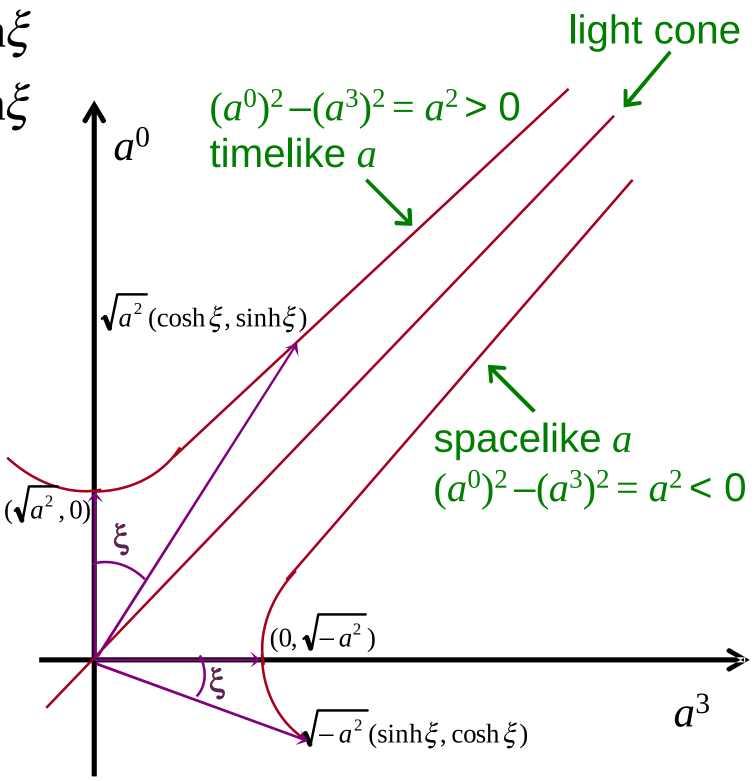
which map the velocity range $-1 \le v/c \le 1$ into the rapidity range $-\infty \le \xi \le \infty$. The product of two transformations is

$$v_3 = c \tanh \xi_3 = \frac{c(\tanh \xi_1 + \tanh \xi_2)}{(1 + \tanh \xi_1 \tanh \xi_2)} = c \tanh(\xi_1 + \xi_2); \quad \xi_3 = \xi_1 + \xi_2$$

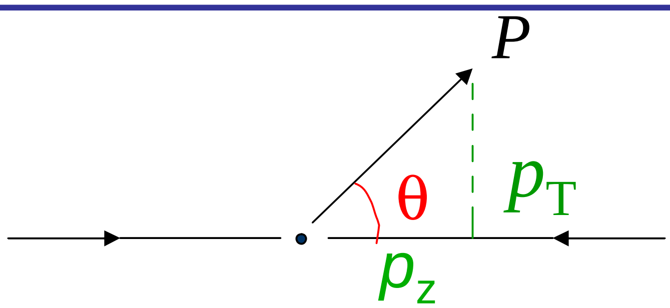
hence rapidities are additive under parallell Lorentz transformations. If v is replaced by ξ in general formula

$$a^0 = a'^0 \cosh \xi + a'^3 \sinh \xi$$
$$a^3 = a'^0 \sinh \xi + a'^3 \cosh \xi$$

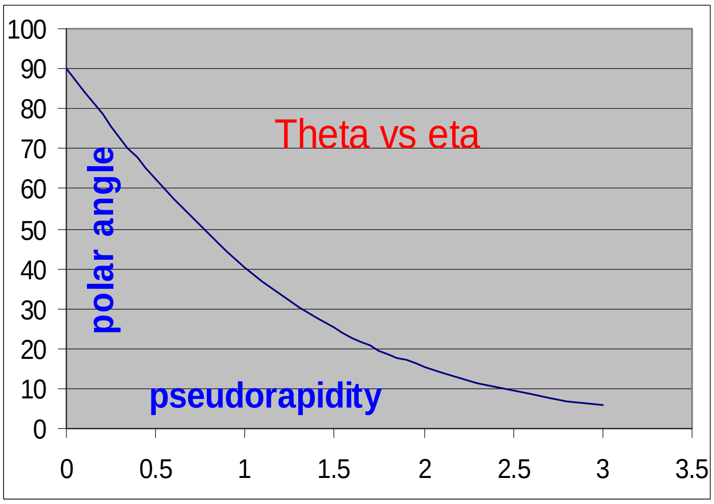
The transformation above leaves the hyperbolas $(a^0)^2 - (a^3)^2$ = constant, i.e. invariant. Rapidity ξ corresponds to an rotation angle in the a^0–a^3 plane (hence the additivity).



Pseudorapidity



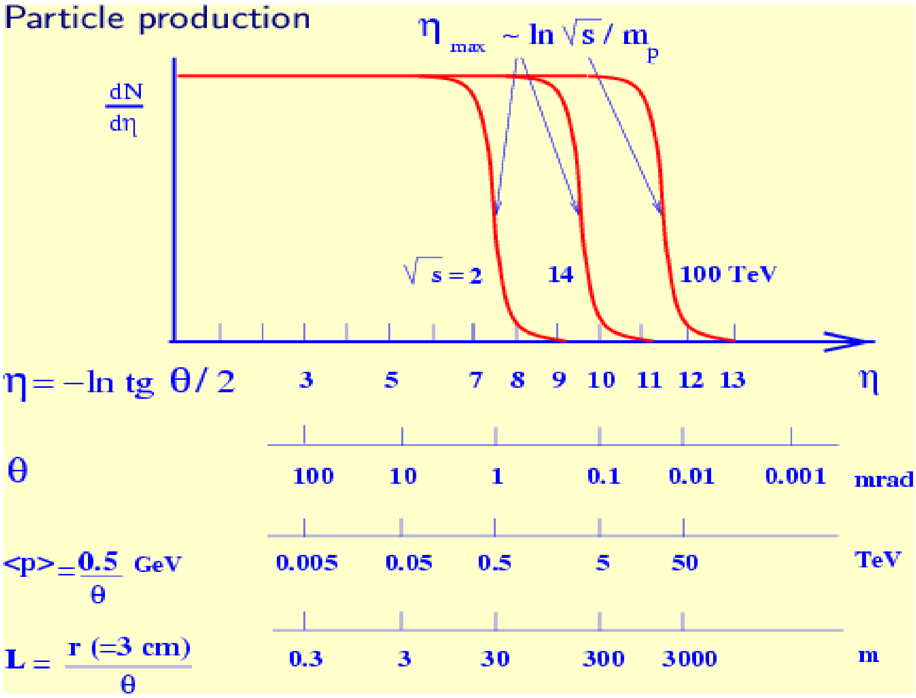
rapidity $\xi = \ln[(E+p_z)/(E-p_z)]/2$ becomes pseudorapidity $\eta = -\ln(\tan(\theta/2))$ if particle masses are neglected ($P,E \gg m$).



$\theta = 90^o \rightarrow \eta = 0$

$\theta = 10^o \rightarrow \eta \approx 2.4$

$\theta = 170^o \rightarrow \eta \approx -2.4$



Pseudospherical coordinates

An arbitrary four-vector a can be parametrized by the parameters of an approriate standard form just like the three-dimensional spherical coordinates give a vector:

$$\bar{a} = A(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad d\Omega = d\cos\theta\, d\phi$$

These "pseudospherical" coordinates are usually defined by (ξ, θ, ϕ) & sometimes by (ξ, ζ, ϕ). ζ is a hyperrotation in the t–xy plane leaving the z component constant.

$$a^0 = \mathrm{A}\cosh\zeta \quad a^1 = \mathrm{A}\sinh\zeta\cos\phi \quad a^2 = \mathrm{A}\sinh\zeta\sin\phi \quad dg = d\cosh\zeta\, d\phi$$

ranges: $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $-\infty \le \xi \le \infty$, $0 \le \zeta \le \infty$.

If a ($a^2 > 0$) is timelike then

$$a = \sqrt{a^2}(\cosh\xi, \sinh\xi\sin\theta\cos\phi, \sinh\xi\sin\theta\sin\phi, \sinh\xi\cos\theta) \quad \text{or}$$

$$a = \sqrt{a^2}(\cosh\xi\cosh\zeta, \cosh\xi\sinh\zeta\cos\phi, \cosh\xi\sinh\zeta\sin\phi, \sinh\xi)$$

If a ($a^2 < 0$) is spacelike then

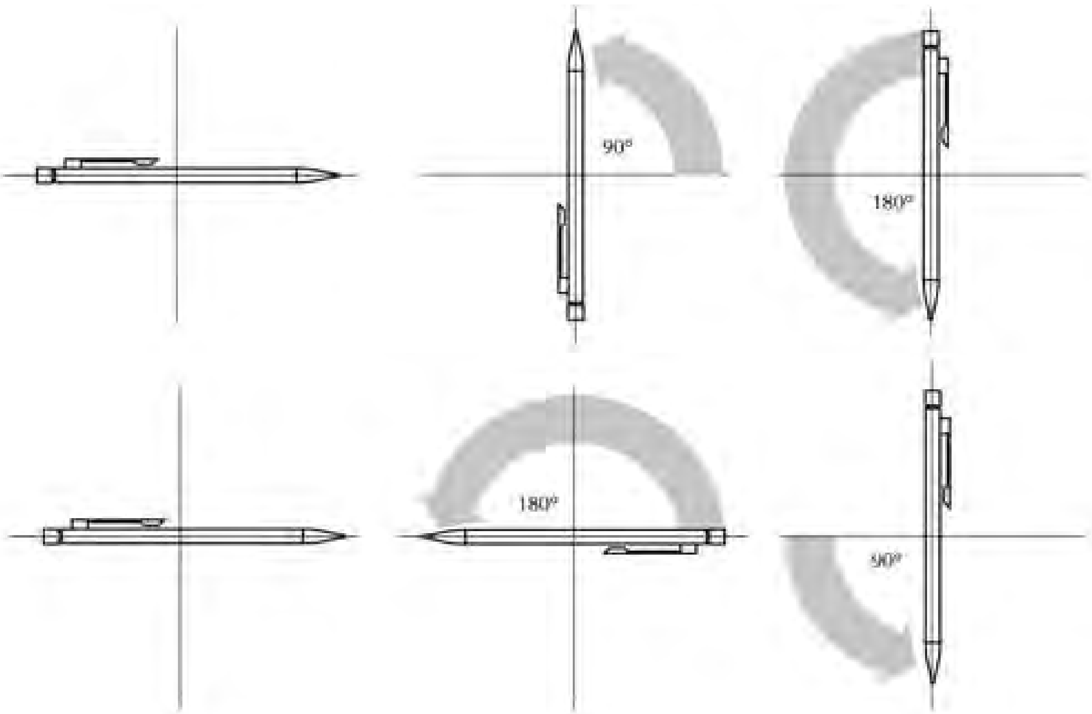
$$a = \sqrt{-a^2}(\sinh\xi, \cosh\xi\sin\theta\cos\phi, \cosh\xi\sin\theta\sin\phi, \cosh\xi\cos\theta) \quad \text{or}$$

$$a = \sqrt{-a^2}(\sinh\xi\cosh\zeta, \sinh\xi\sinh\zeta\cos\phi, \sinh\xi\sinh\zeta\sin\phi, \cosh\xi)$$

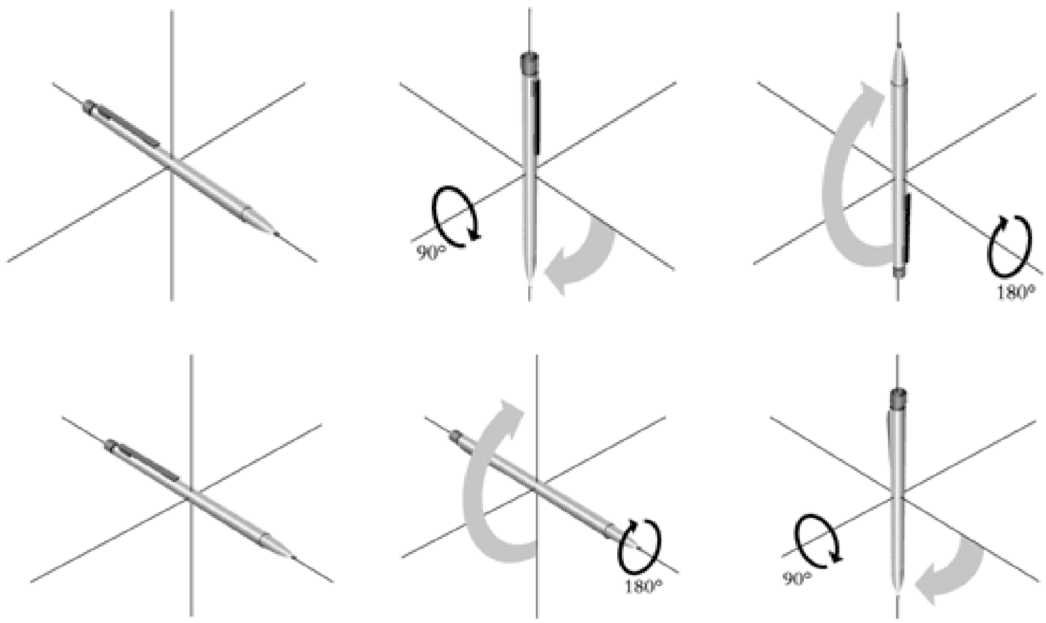
$\Omega \equiv (\cos\theta, \phi)$ and $g \equiv (\cosh\zeta, \phi)$ parametrize Lorentz transformations leaving a^0 and a^3 invariant, respectively. They constitute a O(3) and a O(1,2) group, respectively. The elements of O(1,2) leave $\mathrm{A}^2 = (a^0)^2 - (a^1)^2 - (a^2)^2 = a^2 - (a^3)^2$ invariant just as the elements of O(3) leave $\bar{a}^2 = (a^1)^2 + (a^2)^2 + (a^3)^2 = -a^2 + (a^0)^2$ invariant. This is now sufficient.

NB! O(3) is a non-Abelian group, i.e. the order of two rotations ("generators") plays a difference (see gauge theory).

Abelian group: rotation on a plane



Non-abelian group: rotation in space



Pseudospherical coordinates

Lets now give the matrices of the Lorentz transformation:

$$L(\xi,\theta,\phi) = R_z(\phi)R_y(\theta)L_z(\xi) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix}$$

$$L(\xi,\zeta,\phi) = R_z(\phi)L_x(\zeta)L_z(\xi) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\varsigma & \sinh\varsigma & 0 & 0 \\ \sinh\varsigma & \cosh\varsigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix}$$

Applied to standard vectors $\sqrt{a^2}(1,0,0,0)$ or $\sqrt{-a^2}(0,0,0,1)$, the solutions of previous page should be obtained. Notice that even in the rest frame there will be a non-zero value a direct along z axis, rapidity creates a boost along z.

The corresponding differential volume elements are

$$d^4a = da^0da^1da^2da^3 = d(a^2)A^2dAd\Omega/2a^0; \qquad A = |\bar{a}|$$
$$d^4a = da^0da^1da^2da^3 = d(a^2)\mathrm{A}^2\,d\,\mathrm{A}\,dg/2a^3; \qquad \mathrm{A}^2 = a^2 - (a^3)^2$$

The differential d^4a is of course Lorentz invariant.

$$NB! \quad \int d\Omega = 4\pi \qquad \int dg = \infty$$

Four-velocity and four-momentum

The three-velocity is defined as $\bar{v} = d\bar{x}/dt$ but since t is not an invariant, so $\bar{v}$ doesn't transform like the space component of a four-vector. To construct a velocity four-vector, one needs to find some invariant variable related to time. A natural choice is proper time τ:

$$d\tau^2 = c^{-2}dx^2 \quad \Rightarrow \quad d\tau = dt\sqrt{1 - c^{-2}\left(d\bar{x}/dt\right)^2} = dt/\gamma(\bar{v})$$

In the rest frame, $t = \tau$ that explains the term proper time. The factor γ causes time dilation. Define four-velocity as:

$$u = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma(\bar{v})\frac{dx}{dt} = \gamma(\bar{v})\left(c, \bar{v}\right) \quad \Rightarrow \quad u^2 = c^2$$

u is a four-vector so u^2 is an (timelike) invariant. The basic kinematics four-vector is the four-momentum defined as:

$$p = mu = m\gamma(\bar{v})\left(c, \bar{v}\right) = \left(E/c, \bar{p}\right) \quad \Rightarrow \quad p^2 = m^2c^2 = \left(E/c\right)^2 - \bar{p}^2$$

from this equation one can obtain the following relations:

$$\bar{\beta} \equiv \bar{v}/c = \bar{p}c/E \qquad \gamma(\bar{v}) = E/mc^2 \qquad \bar{\beta}\gamma(\bar{v}) = \bar{p}/mc$$

the Lorentz transformation of the four-momentum is:

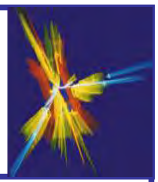
$$p_x' = p_x \qquad p_z' = \gamma(p_z - vE/c^2)$$

$$p_y' = p_y \qquad E' = \gamma(E - vp_z)$$

for the rapidity ξ this leads to the following expression:

$$\xi = \text{arctanh}\,(v/c) = 1/2\ln\{(1 + v/c)/(1 - v/c)\}$$

$$\xi = \text{arcsinh}(v\gamma/c) = \ln\{\gamma(1 + v/c)\}$$



Natural units:

Special relativity: $E^2 = \vec{p}^2 c^2 + m^2 c^4$

Have 3 fundamental units: length L, time T and energy E

2 natural constants: $c = 3.0 \cdot 10^8$ m/s, $\hbar = 6.6 \cdot 10^{-25}$ GeVs

Set c = 1 = [L] / [T] ⇒ [T] = [L] (applied e.g. in four-vectors)

Also set $\hbar$ = 1 = [E] · [T] ⇒ [L] = [T] = 1/ [E] (= GeV^{-1})

One degree-of-freedom left so choose [E] = GeV

1 GeV is a tiny portion of energy. 1 GeV = $1.6 \cdot 10^{-10}$ J

$m_{\text{bee}} \approx$ 1 g = $5.6 \cdot 10^{23}$ GeV/c^2

$v_{\text{bee}} \approx$ 1 m/s → E_{bee} = $0.5 \cdot 10^{-3}$ J = $3.1 \cdot 10^6$ GeV

⇔ E_{LHC} = 14000 GeV (collision energy at start 2007)

To rehabilitate LHC…

Stored energy/beam:

$3.2 \cdot 10^{14}$ protons * 7000 GeV ≈ $3.6 \cdot 10^8$ J

this corresponds to a

$m_{\text{truck}} \approx$ 100 T

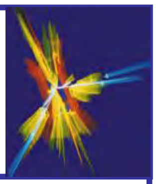
$v_{\text{truck}} \approx$ 300 km/h

Now $E^2 = \vec{p}^2 + m^2$ ⇒ [E] = [P] = [m] = 1 GeV

Define $\beta = \frac{v}{c}$ $(0 \le \beta < 1)$ $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $(1 \le \gamma < \infty)$

e.g. $\beta = P / E$, $\gamma = E / m$, and <lifetime> = $\beta \cdot \gamma \cdot c \cdot \tau$

Units and conventions



The choice of energy variable (given in eV) is not unique:

- kinetic energy, $T = E - m$, is used in the domain where the rest energy > the kinetic energy (e.g. nuclear physics).
- total energy, E, is sometimes used at high energies.
- **momentum,** P, is normally used at collider experiments (magnets select by momentum, experiments measure dito !!)

Useful relations to remember by heart are:

$\hbar c$ = 197.33 MeV fm $\qquad$ $\hbar = 6.5822 \cdot 10^{-22}$ MeV s

e.g. E of a photon: $\quad E = hc/\lambda \approx 1240/\lambda[\mathrm{fm}]\ \mathrm{MeV}$

Physically p is always forward timelike, $E > 0$, but one can also formally consider backward timelike four-momenta. If $p=(E,\bar{p})$ is backward timelike, $-p=(-E,-\bar{p})$ is then the four-momentum of a physical particle (QM $\Rightarrow$ the antiparticle).

Scalar products $p_i \cdot p_j = E_i E_j - \bar{p}_i \cdot \bar{p}_j$ are invariant by definition. Some scalar products are very commonly used like for example the two particle invariant mass squared:

$$s_{12} = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\bar{p}_1 + \bar{p}_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2$$

another one is the invariant momentum transfer squared

$$t_{12} = (p_1 - p_2)^2 = (E_1 - E_2)^2 - (\bar{p}_1 - \bar{p}_2)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2$$

since m_1 and m_2 are constants, s_{12} and t_{12} will attain their extreme values simultaneously, i.e. when $\bar{v}_1 = \bar{v}_2$, hence

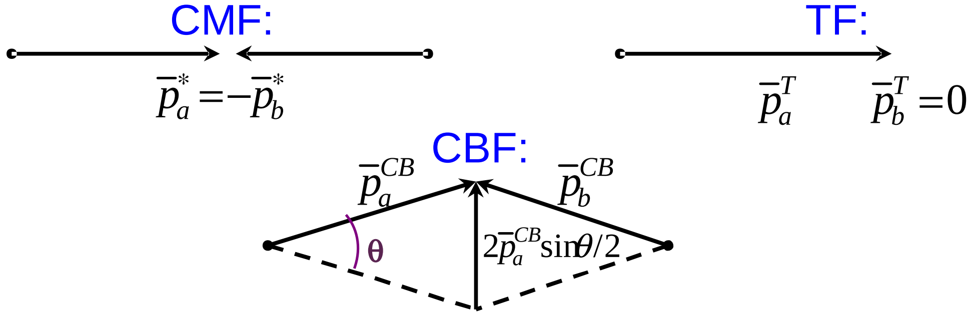
$$p_1 \cdot p_2 \ge m_1 m_2 \quad (\text{found by going to } \bar{v}_1 = \bar{v}_2 = 0) \quad \Rightarrow$$

$$s_{12} \ge (m_1 + m_2)^2 \qquad t_{12} \le (m_1 - m_2)^2 \quad (= \text{when } v_1 = v_2 \text{ in any frame})$$

Useful frames for scattering processes

Let's introduce some frames, defined by the initial state of a scattering process. In a two-particle process, particles *a* and *b* with four-momenta $p_a = (E_a, \bar{p}_a)$ & $p_b = (E_b, \bar{p}_b)$ interact.

1. Laboratory frame (LF) is defined as the frame in which the experiment is carried out and all energies and momenta measured. This is the primary frame from which quantities (usually denoted by an index L) are transformed to other systems.
2. Centre-of-mass frame (CMF) is defined as the frame in which $\bar{p}_a^* + \bar{p}_b^* = 0$. The CMF quantities are usually denoted by an asterisk or an index CM.
3. Target (TF) and beam frames (BF) are defined as the frames in which $\bar{p}_b^T = 0$ and $\bar{p}_a^B = 0$ respectively. Many experiments are fixed target experiments, i.e. LF = TF. Kinematically TF and BF are equivalent.
4. Colliding beam frame (CBF) is sometimes defined as the frame in which two beams collide (usually the same as LF) with an angle θ. If the momenta of the colliding particles is equal & θ = 0 then CBF = CMS.



CBF example with equal momenta & a collision angle θ

Lorentz transformations between frames

Consider first Lorentz transformations between the CMF and TF frames. Initial state can be expressed as follows:

$$p_a^* = (E_a^*, 0, 0, P_a^*) \qquad p_a^T = (E_a^T, 0, 0, P_a^T)$$
$$p_b^* = (E_b^*, 0, 0, -P_a^*) \qquad p_b^T = (m_b, 0, 0, 0)$$

where the direction of motion has been chosen as the z axis. The Lorentz transformation equations are now:

$$P_a^* = \gamma^{CM}(P_a^T - v^{CM} E_a^T) \qquad v^{CM} \text{ is velocity}$$
$$E_a^* = \gamma^{CM}(E_a^T - v^{CM} P_a^T) \qquad \text{of CMF in TF}$$

need to determine v^{CM}. Total energy and momentum of a group of particles is $E_{tot}, \bar{p}_{tot}$ in some reference frame then

$$\bar{v}_{tot} = \bar{p}_{tot} / E_{tot} \qquad \gamma_{tot} = E_{tot} / m_{tot} \qquad \gamma_{tot}\bar{v}_{tot} = \bar{p}_{tot} / m_{tot}$$

where $m_{tot} = \sqrt{E_{tot}^2 - \bar{p}_{tot}^2} = \sqrt{s}$ is the invariant mass of the group of particles. For a 2-particle system this becomes:

$$s \equiv s_{ab} = (E_a + E_b)^2 - (\bar{p}_a + \bar{p}_b)^2 = (E_a^T + m_b)^2 - (P_a^T)^2 = m_a^2 + m_b^2 + 2m_b E_a^T$$

Now the CMF-TF relation can be expressed as:

$$v^{CM} = P_a^T \big/ \left(E_a^T + m_b\right) \qquad \gamma^{CM} = \left(E_a^T + m_b\right) \big/ \sqrt{s} \qquad \gamma^{CM} v^{CM} = P_a^T \big/ \sqrt{s}$$

inserting these into the Lorentz transformation equations

$$P_a^* = m_b P_a^T \big/ \sqrt{s} \qquad E_a^* = \left(m_a^2 + m_b E_a^T\right) \big/ \sqrt{s}$$
$$P_b^* = m_b P_a^T \big/ \sqrt{s} = P_a^* \qquad E_b^* = m_b\left(m_b + E_a^T\right) \big/ \sqrt{s}$$

The Lorentz transformations can be done explicitly as above but in more complicated cases this becomes too tedious (and error prone) so instead noninvariants will be expressed in terms of invariants to make algebra easier.

E, *P* in terms of invariants: TF

For the target frame (TF) we have $\overline{p}_b^T = 0$ and $E_b^T = m_b$

$$E_a^T = (s - m_a^2 - m_b^2)/2m_b \quad (P_a^T)^2 = (E_a^T)^2 - m_a^2 = \left\{(s - m_a^2 - m_b^2) - 4m_a^2 m_b^2\right\}/4m_b^2$$

To simplify we introduce a kinematical function λ:

$$\lambda(x,y,z) = (x-y-z)^2 - 4yz = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$
$$= \left\{x - (\sqrt{y} + \sqrt{z})^2\right\}\left\{x - (\sqrt{y} - \sqrt{z})^2\right\}$$
$$= \left\{\sqrt{x} - \sqrt{y} - \sqrt{z}\right\}\left\{\sqrt{x} + \sqrt{y} + \sqrt{z}\right\}\left\{\sqrt{x} - \sqrt{y} + \sqrt{z}\right\}\left\{\sqrt{x} + \sqrt{y} - \sqrt{z}\right\}$$

λ is invariant under all permutations of its arguments (see above). λ is sometimes called the triangle function since $\sqrt{-\lambda(x,y,z)}/4$ is the area of a triangle with sides $\sqrt{x}, \sqrt{y}$ and $\sqrt{z}$

for TF momentum we get: $P_a^T = \sqrt{\lambda(s, m_a^2, m_b^2)}/2m_b$

now: $\lambda(s, m_a^2, m_b^2) = \left\{s - (m_a + m_b)^2\right\}\left\{s - (m_a - m_b)^2\right\}$

thus P_a^T is real if: $\sqrt{s} \geq m_a + m_b$

the threshold value $m_a + m_b$ is the smallest value $\sqrt{s}$ can attain. The same threshold also appears if one writes the kinetic energy T_a of a particle in terms of s.

$$T_a = E_a^T - m_a = \left\{s - (m_a + m_b)^2\right\}/2m_b$$

For λ there are some cases with special interest:

$$\lambda(x,y,y) = x(x-4y) \qquad \lambda(x,y,0) = (x-y)^2$$

the first special case is relevant in e.g. the case where there are two particles with the same mass that are scattering and the second e.g. in the case one of the scattering particles is a massless particle, e.g. a photon.

E, P in terms of invariants: CMF & CBF

For the centre-of-mass frame (CMF):

$$P_a^* = P_b^* = P^* \qquad \sqrt{s} = E_a^* + E_b^* \qquad \Rightarrow$$

$$\sqrt{s} = \sqrt{(P^*)^2 + m_a^2} + \sqrt{(P^*)^2 + m_b^2}$$

$\sqrt{s}$ is equal to the total energy in CMF. One obtains the following expression for CMF energy and momentum:

$$E_a^* = (s + m_a^2 - m_b^2) / 2\sqrt{s}$$

$$P^* = \sqrt{\lambda(s, m_a^2, m_b^2)} / 2\sqrt{s}$$

$$E_b^* = \sqrt{s} - E_a^* \quad \Rightarrow \quad E_b^* = (s - m_a^2 + m_b^2) / 2\sqrt{s}$$

If we return to the transformation between CMF & TF, we can expand the expression for v^{CM} & γ^{CM} using the Taylor expansion of the square root of (1+ε) & 1/(1+ε), an approximation valid for the case where $s \gg m_a^2$, m_b^2.

$$v^{CM} = \frac{P_a^T}{(E_a^T + m_b)} = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{(s - m_a^2 + m_b^2)} \approx 1 - \frac{2m_b^2}{s} + O(s^{-2})$$

$$\gamma^{CM} = \frac{(E_a^T + m_b)}{\sqrt{s}} = \frac{(s - m_a^2 + m_b^2)}{2m_b\sqrt{s}} \approx \frac{\sqrt{s}}{2m_b} + O(s^{-1/2})$$

$$v^{CM}\gamma^{CM} = \frac{P_a^T}{\sqrt{s}} = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b\sqrt{s}} \approx \frac{\sqrt{s}}{2m_b}\left(1 - \frac{m_a^2}{s} - \frac{m_b^2}{s} + O(s^{-2})\right)$$