Decay of one particle into three

(E.Byckling & K.Kajantie)

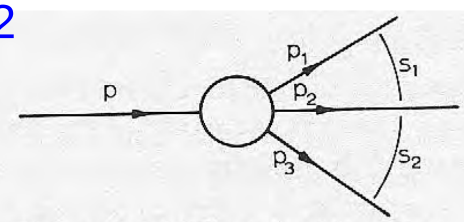

Figure V.1.1 Three-particle decay $p \to p_1 + p_2 + p_3$ with invariant variables s_1 and s_2

Since a decay process $p \to p_1 + p_2 + p_3$ is related by crossing to 2 → 2 scattering, the number of invariant variables must be the same, namely two. Let's consider first invariant & then non-invariant variables for the process 1 → 3.

As invariant variables, it is convient to choose s, t & u as in 2 → 2 scattering. To avoid mixup let's change notation

$$\text{new invariants}: \begin{cases} s_{12} \equiv s_1 = (p_1 + p_2)^2 = (p - p_3)^2 \\ s_{23} \equiv s_2 = (p_2 + p_3)^2 = (p - p_1)^2 \\ s_{31} \equiv s_3 = (p_3 + p_1)^2 = (p - p_2)^2 \end{cases}$$

$$\text{their common relation}: s_1 + s_2 + s_3 = s + m_1^2 + m_2^2 + m_3^2$$

Non-invariant variables are three-momenta & angles. To define them one has to specify a Lorentz frame. The most common one correspond to 2 → 2 scattering CMF & TF.

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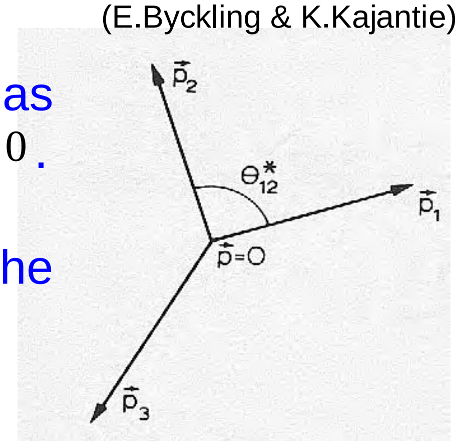


The rest frame of the decaying particle or overall CMF is defined as the frame in which $\bar{p} = \bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 0$. This is the analogue of TF in 2 → 2 scattering in the sense that one of the external momenta is taken to be at rest. Quantities in this frame are denoted by an asterisk.

1→3: non-invariant variables

Expressions for energies and momenta can immediately be derived using s_i (i = 1...3) definitions in frame $p = (\sqrt{s}, \overline{0})$.

$$E_1^* = \frac{s + m_1^2 - s_2}{2\sqrt{s}} \qquad E_2^* = \frac{s + m_2^2 - s_3}{2\sqrt{s}} \qquad E_3^* = \frac{s + m_3^2 - s_1}{2\sqrt{s}}$$

$$P_1^* = \frac{\sqrt{\lambda(s, m_1^2, s_2)}}{2\sqrt{s}} \qquad P_2^* = \frac{\sqrt{\lambda(s, m_2^2, s_3)}}{2\sqrt{s}} \qquad P_3^* = \frac{\sqrt{\lambda(s, m_3^2, s_1)}}{2\sqrt{s}}$$

Note the logic: E_1^* is obtained by considering the two-particle decay $p \to p_1 + (p_2 + p_3)$ with final state masses m_1 & $\sqrt{s_2}$. The angles between the momentum vectors follow from expansions of the s_i (i = 1...3) definitions, e.g.

$$s_1 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1^* E_2^* - 2P_1^* P_2^* \cos\theta_{12}^* \quad \Rightarrow$$

$$\cos\theta_{12}^* = \left.\frac{\overline{p}_1 \cdot \overline{p}_2}{P_1 P_2}\right|_{\overline{p}=0} = \frac{(s + m_1^2 - s_2)(s + m_2^2 - s_3) + 2s(m_1^2 + m_2^2 - s_1)}{\sqrt{\lambda(s, m_1^2, s_2)}\sqrt{\lambda(s, m_2^2, s_3)}}$$

$\sin\theta_{12}^*$ is related to corresponding symmetric Gram determinant Δ_3 in CMF: $\sin^2\theta_{12}^* = \left.\frac{|\overline{p}_1 \times \overline{p}_2|^2}{P_1^2 P_2^2}\right|_{\overline{p}=0} \Rightarrow$

$$\sin^2\theta_{12}^* = \frac{\Delta_3(-p_1, p, p_2)}{s(P_1^*)^2(P_2^*)^2} = \frac{-4sG(s_2, s_1, m_3^2, m_1^2, s, m_2^2)}{\lambda(s, m_1^2, s_2)\lambda(s, m_2^2, s_3)}$$

There are three possible rest frames of a produced 2–particle system, corresponding to s, t and u channel CMF-systems in 2 → 2 scattering. They are following:

$$\overline{p}_1 + \overline{p}_2 = \overline{p} - \overline{p}_3 = 0 \qquad \overline{p}_2 + \overline{p}_3 = \overline{p} - \overline{p}_1 = 0 \qquad \overline{p}_3 + \overline{p}_1 = \overline{p} - \overline{p}_2 = 0$$

1→3: non-invariant variables

We denote quantities in these frames by superscripts R12, R23, R31, respectively (R for rest). It is also sufficient to consider one of these, say R23, since equations refering to other frames can be obtained by cyclic permutations.

(E.Byckling & K.Kajantie)

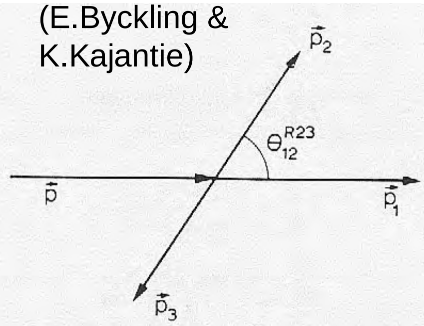


Figure V.1.3 Rest frame of the system formed by the particles 2 and 3 ($\mathbf{p}_2 + \mathbf{p}_3 = 0$)

R23 energies & momenta in terms of invariants found by expanding $s_2 = (p_2 + p_3)^2$ in R23 ⇔ frame $p_2 + p_3 = (\sqrt{s_2}, \bar{0})$.

$$E_{(1)}^{R23} = \left(s \pm s_2 - m_1^2\right)/2\sqrt{s_2} \qquad E_{2(3)}^{R23} = \left(s_2 \pm m_2^2 - \pm m_3^2\right)/2\sqrt{s_2}$$

$$P^{R23} = P_1^{R23} = \frac{\sqrt{\lambda(s, s_2, m_1^2)}}{2\sqrt{s_2}} \qquad P_2^{R23} = P_3^{R23} = \frac{\sqrt{\lambda(s_2, m_2^2, m_3^2)}}{2\sqrt{s_2}}$$

Only one angle in R23 is essential for the decay, namely θ_{12}^{R23}, that can be obtained by e.g. expanding s_1

$$s_1 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1^{R23}E_2^{R23} - 2P_1^{R23}P_2^{R23}\cos\theta_{12}^{R23} \quad \Rightarrow$$

$$\cos\theta_{12}^{R23} = \left.\frac{\bar{p}_1 \cdot \bar{p}_2}{P_1 P_2}\right|_{\bar{p}_2 = -\bar{p}_3} = \frac{(s - s_2 - m_1^2)(s_2 + m_2^2 - m_3^2) + 2s_2(m_1^2 + m_2^2 - s_1)}{\sqrt{\lambda(s, s_2, m_1^2)}\sqrt{\lambda(s_2, m_2^2, m_3^2)}}$$

$\sin\theta_{12}^{R23}$ is ∝ to Δ_3 in R23: $\sin^2\theta_{12}^{R23} = \left.\frac{|\bar{p}_1 \times \bar{p}_2|^2}{P_1^2 P_2^2}\right|_{\bar{p}_2 = -\bar{p}_3} \Rightarrow$

$$\sin^2\theta_{12}^{R23} = \frac{\Delta_3(-p_1, p, p_2)}{s_2 (P_1^{R23})^2 (P_2^{R23})^2} = \frac{-4 s_2 G(s_2, s_1, m_3^2, m_1^2, s, m_2^2)}{\lambda(s, s_2, m_1^2)\lambda(s_2, m_2^2, m_3^2)}$$

For many purposes the kinematics is simpler in the R12, R23 & R31 frames than in overall CMF, since for decay 1 → 3, these frames are similar to CMF in 2 → 2 scattering.

Dalitz plot

The Dalitz plot is defined as the physical region of $p \rightarrow p_1 + p_2 + p_3$ in the $s_1 s_2$ plane. More generally, can be defined as the physical region in terms of any variables related to s_1 & s_2 by a linear transformation with constant Jacobian e.g. any pair s_i, s_j or any pair E_i^*, E_j^*, where i & j = 1...3.

The Dalitz plot is given by all points in the $s_1 s_2$ plane that satisfies the following equation: $G(s_2, s_1, m_3^2, m_1^2, s, m_2^2) \leq 0$

(E.Byckling & K.Kajantie)

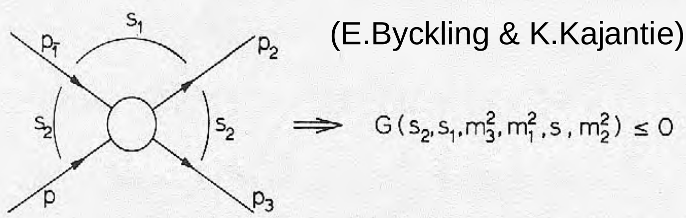


Figure V.2.1 A direct procedure giving the physical region in the $s_1 s_2$ plane

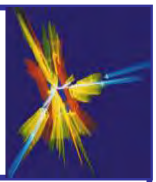
The G here is the same as in the expressions for $\sin\theta_{12}^{R23}$ and $\sin\theta_{12}^*$. The equality gives the boundary in the Dalitz plot. This can be obtained e.g. by solving s_1 in terms of s_2.

$$s_1^{\pm} = m_1^2 + m_2^2 - \frac{\left\{(s_2 + m_1^2 - s)(s_2 + m_2^2 - m_3^2) - \pm\sqrt{\lambda(s_2, s, m_1^2)}\sqrt{\lambda(s_2, m_2^2, m_3^2)}\right\}}{2s_2}$$

The equation giving s_2 in terms of s_1 is obtained from the above by the exchange $p_1 \leftrightarrow p_3$; p, p_2 unchanged. Both give, of course, the same curve. By requiring the √ to be real, one gets the physical region in s_2. Cyclic symmetry implies that also following conditions have to be satisfied:

$$(m_1 + m_2)^2 \leq s_1 = (p_1 + p_2)^2 \leq (\sqrt{s} - m_3)^2$$
$$(m_2 + m_3)^2 \leq s_2 = (p_2 + p_3)^2 \leq (\sqrt{s} - m_1)^2$$
$$(m_3 + m_1)^2 \leq s_3 = (p_3 + p_1)^2 \leq (\sqrt{s} - m_2)^2$$

Dalitz plot



To determine the phase space density & obtain as well the condition for the boundary of the Dalitz plot directly, let's consider the phase space integral:

$$R_3(s) = \int \prod_{i=1}^{3} \frac{d^3\overline{p}_i}{2E_i} \delta^3(\overline{p} - \overline{p}_1 - \overline{p}_2 - \overline{p}_3)\delta(\sqrt{s} - E_1 - E_2 - E_3)$$

Integrate first over $\overline{p}_2$ in the rest frame $\overline{p} = 0 \quad \Rightarrow$

$$R_3(s) = \int \frac{d^3\overline{p}_1^* d^3\overline{p}_3^*}{8E_1^* E_2^* E_3^*} \delta(\sqrt{s} - E_1^* - E_2^* - E_3^*), \quad \text{where}$$

$$E_2^{*2} = |\,\overline{p}_1^* + \overline{p}_3^*\,|^2 + m_2^2 = P_1^{*2} + P_3^{*2} + 2P_1^* P_3^* \cos\theta_{13}^* + m_2^2$$

Write further $\quad d^3\overline{p}_1\, d^3\overline{p}_3 = P_1 E_1 dE_1 d\Omega_1\, P_3 E_3 dE_3 d\cos\theta_{13} d\phi_3$

The δ–function containing energies can be used to integrate over $\cos\theta_{13}$* ($dE_2^* / d\cos\theta_{13}^* = P_1^* P_3^* / E_2^*$) giving:

$$R_3(s) = \int \frac{dE_1^* dE_3^* d\Omega_1^* d\phi_3^* P_1^* E_1^* P_3^* E_3^*}{8E_1^* E_2^* E_3^* (P_1^* P_3^* / E_2^*)} \Theta(1 - \cos^2\theta_{13}^*)$$

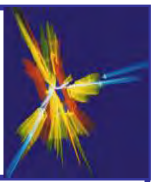
$$= \int \frac{dE_1^* dE_3^* d\Omega_1^* d\phi_3^*}{8} \Theta(1 - \cos^2\theta_{13}^*)$$

Here the Θ–function restricts $\cos\theta_{13}$* to physical values only.

The variables E_1* & E_3* are linearly connected to s_1 & s_2 with the Jacobian $\partial(E_1^*, E_3^*) / \partial(s_2, s_1) = 1 / 4s$, thus

$$R_3(s) = \frac{1}{32s} \int ds_1 ds_2 d\Omega_1^* d\phi_3^* \Theta\{-G(s_2, s_1, m_3^2, m_1^2, s, m_2^2)\}$$

NB! the $\cos\theta_{13}$*–condition is exchanged to a G–function condition, obtained by algebra from the E_2*–condition in the δ–function. Analogous forms of $R_3(s)$ with the pairs s_2, s_3 & s_3, s_1 obtained by cyclic permutations of indices.



The solid angle Ω_1* describes the $\bar{p}_1$–orientation in CMF & ϕ_3*, the rotation of the entire momentum configuration about some axis. Integrating over Ω_1* & ϕ_3*, we obtain:

$$\text{The phase space distribution}: \frac{d^2R_3}{ds_1 ds_2} = \frac{\pi^2}{4s} \quad (=\text{constant})$$

In other words, if data of a three-particle decay is shown as points in a Dalitz plot, the density of points ∝ (matrix element)^2. Any structure is thus easily evident. This is why the Dalitz plot is so famous & used often. Note that this result is strictly only valid for three-particle decays.

$$\text{Further} \quad \frac{\mathrm{dR}_3}{\mathrm{ds}_2} = \frac{\pi^2}{4s}\int_{s_1^-}^{s_1^+} ds_1 = \frac{\pi^2}{4ss_2}\sqrt{\lambda(s_2,s,m_1^2)}\sqrt{\lambda(s_2,m_2^2,m_3^2)} \quad \Rightarrow$$

For the total volume of the three-particle phase space:

$$\mathrm{R}_3(s) = \frac{\pi^2}{4s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{ds_2}{s_2}\sqrt{\lambda(s_2,s,m_1^2)}\sqrt{\lambda(s_2,m_2^2,m_3^2)}$$

The √–factor is 4th order in s_2 & lead to elliptic functions. Explicit solutions exists only for some special cases.

Especially interesting are the extremes, $s \to$ threshold = $(m_1+m_2+m_3)^2$ (decay products non-relativistic, NR) or $s \to \infty$ (ultra-relativistic, UR). Latter is obtained by setting all m_i = 0.

$$\mathrm{R}_3^{\mathrm{UR}}(s) = \frac{\pi^2 s}{8} \qquad \mathrm{R}_3^{\mathrm{NR}}(s) = \frac{\pi^3\sqrt{m_1 m_2 m_3}}{2(m_1+m_2+m_3)^{\frac{3}{2}}}(\sqrt{s}-m_1-m_2-m_3)^2$$

Dalitz plot

Let's next consider the boundary for some special cases:

– two or three masses vanishes. There are three distinct cases: $m_2 = m_3 = 0$, $m_1 = m_3 = 0$ and $m_1 = m_2 = m_3 = 0$.

$$G(s_2, s_1, 0, m_1^2, s, 0) = s_2\{s_1(s_1 + s_2 - s) - m_1^2(s_1 - s)\} \leq 0$$

$$G(s_2, s_1, 0, 0, s, m_2^2) = (s_1 s_2 - s m_2^2)(s_1 + s_2 - s - m_2^2) \leq 0$$

$$G(s_2, s_1, 0, 0, s, 0) = s_1 s_2 (s_1 + s_2 - s) = -s_1 s_2 s_3 \leq 0$$

The case $m_1 = m_2 = 0$ follows by symmetry from $m_2 = m_3 = 0$. All these boundaries factorize. The plots below are ≈ true in the case the masses are small compared to $\sqrt{s}$. In particular, when $s \to \infty$, the Dalitz plot is a triangle.

(E.Byckling & K.Kajantie)

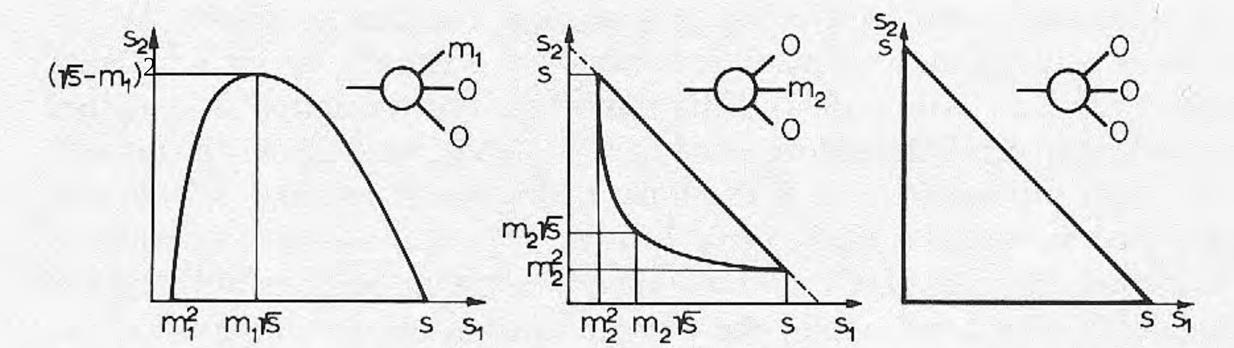


Figure V.2.2 The schematic Dalitz plots for the special mass values shown in the figure

– all masses equal. This was the case discussed by R. Dalitz himself in 1953, when he proposed his recipe. He did it in terms of the three CMF kinetic energies & used triangular coordinates. His orginal notation & recipe are not relevant for modern particle physics experiments (see e.g. R. Hagedorn: Relativistic kinematics (W.A. Benjamin 1964)).

Momentum configuration in Dalitz plot

The final state momentum vectors are collinear on the Dalitz plot boundary, as expected. Let's consider overall CMF and see where s_1, s_2 & s_3 attain their maxima and minima. For s_1, $s_1 = (m_1 + m_2)^2$ obtained when $p_1 \cdot p_2 = m_1 m_2$.

$$\Rightarrow \quad \cos\theta_{12} = 1 \quad \wedge \quad P_1/m_1 = P_2/m_2 \,(\Rightarrow \beta_1 = \beta_2)$$

s_1 as small as possible implies maximal possible E_3^*:

$$E_3^* = E_3^{*,\max} = \frac{s + m_3^2 - (m_1 + m_2)^2}{2\sqrt{s}} \qquad \text{(point "A}_1\text{")}$$

The maximal value of s_1, $s_1 = (\sqrt{s} - m_3)^2$ corresponds to $P_3^* = 0$ ($E_3^* = m_3$) or $\overline{p}_1^* = -\overline{p}_2^*$ (point "B$_1$"). The minima & maxima for s_2 & s_3 are obtained by cyclic permulations.

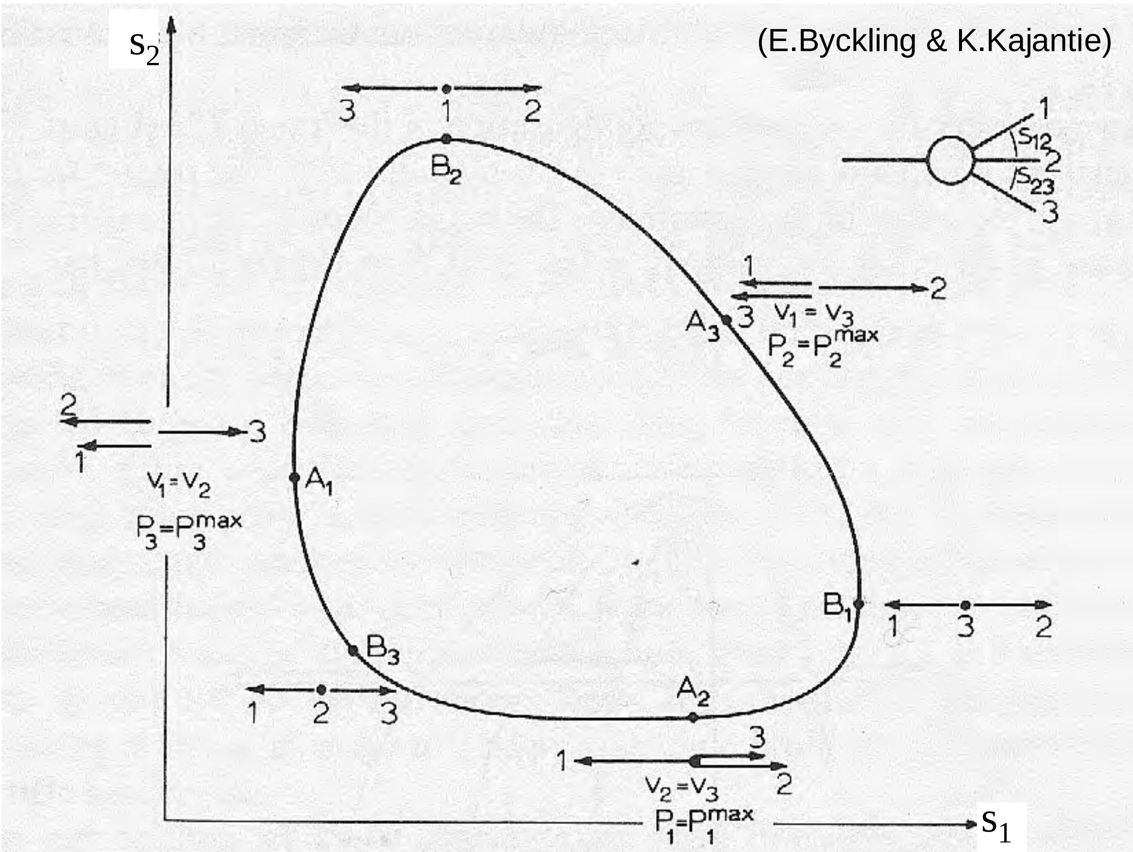


Figure V.3.1 Configurations of momentum vectors in overall CMS on the boundary of the Dalitz plot

$2 \rightarrow 3$ particle scattering

The initial state of $2 \rightarrow 3$ scattering, $p_a + p_b \rightarrow p_1 + p_2 + p_3$, contains in CMF a preferred direction, the incoming beam one $\bar{p}_a^* = -\bar{p}_b^*$. The total number of final state variables is 5, of which the rotation around the beam axis is trivial for spinless particles & hence the number of essential final state variables is 4. It is no longer possible to present data & predictions in fully differential form (would require an intensity plot in 4 dimensions). At best one can plot in 2 dimensions and integrate or fix the remaining variables.

There are several ways to proceed from the original form of three-particle phase space integral $R_3(s)$. Here we will integrate of the δ–functions & then replace stage by stage the remaining 4 non-invariant variables by invariant ones. For defining any angular variable in $2 \rightarrow 3$ scattering, one has to specify a frame & orientations of coordinate axes. On the contrary, the invariant variables can be defined as:

$$s_1 \equiv s_{12} = (p_1 + p_2)^2 = (p_a + p_b - p_3)^2$$
$$s_2 \equiv s_{23} = (p_2 + p_3)^2 = (p_a + p_b - p_1)^2$$
$$t_1 \equiv t_{a1} = (p_a - p_1)^2 = (p_2 + p_3 - p_b)^2$$
$$t_2 \equiv t_{b3} = (p_b - p_3)^2 = (p_1 + p_2 - p_a)^2$$
$$s \equiv s_{ab} = (p_a + p_b)^2 = (p_1 + p_2 + p_3)^2$$

(E.Byckling & K.Kajantie)

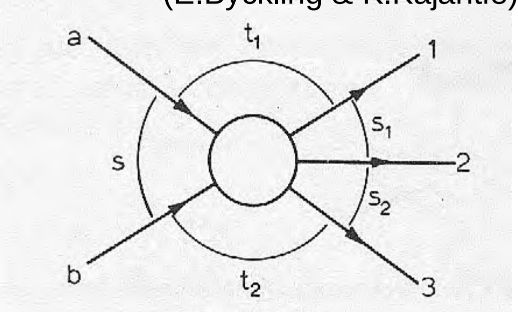


Figure V.4.1

In the phase space of $2 \rightarrow 3$ scattering, s is fixed & the other invariants vary. However, when this condition is relaxed, the invariants defined above are related by crossing under the following cyclic transformation: $s \rightarrow t_1 \rightarrow s_1 \rightarrow s_2 \rightarrow t_2 \rightarrow s$ (i.e. $p_a \rightarrow -p_1 \rightarrow -p_2 \rightarrow -p_3 \rightarrow p_b \rightarrow p_a$).

In addition to the invariants above, one can define 5 other invariants by joining particles that are not adjacent in the diagram. These are t_{a2}, t_{b2}, t_{a3}, t_{b1} & s_{13}. They are linearly dependent on the defined set. The relation between e.g. t_{b2} & the defined set is most easily obtained by drawing the diagram below & applying the relation corresponding to $s + t + u = \Sigma\, m_i^2$ in the lower 2 → 2 scattering vertex.

(E.Byckling & K.Kajantie)

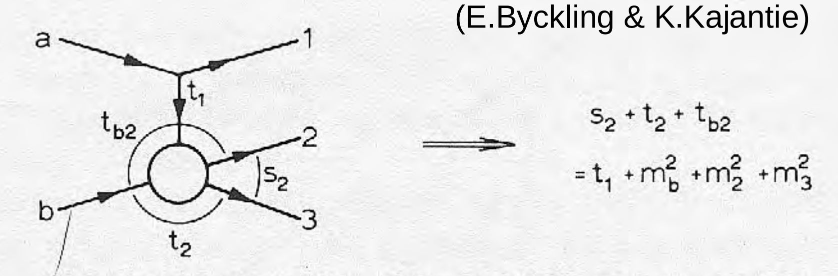
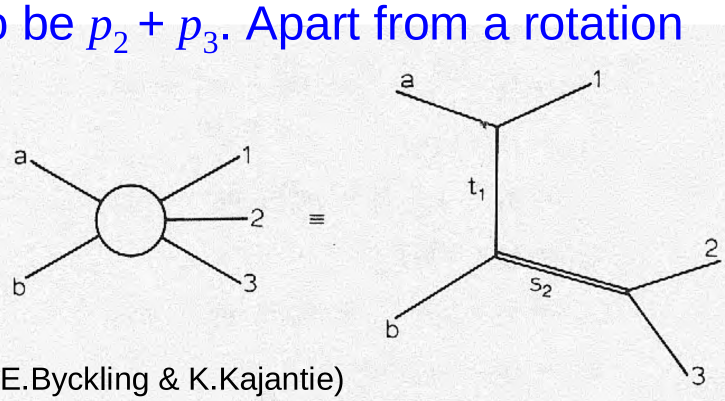


Figure V.4.2 Derivation of the relation between t_{b2} and the standard invariants

Further can the 10 scalar products $p_i \cdot p_j$ be expressed in terms of the defined set of invariants using same recipe.

E.g. $$2\,p_b \cdot p_2 = p_b^2 + p_2^2 - t_{b2} = s_2 + t_2 - t_1 - m_3^2$$

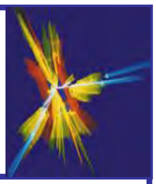
A detailed treatment of the 2 → 3 scattering will be based on the factorization of the phase space integral into two processes 2 → 2 and 1 → 2. We'll choose the 2–particle intermediate system to be $p_2 + p_3$. Apart from a rotation around the beam axis, both the production of the system 2+3 & its decay are described by two variables.



(E.Byckling & K.Kajantie)

Figure V.5.1

Phase space integral



Here the first two are taken to be invariants & the latter two decay angles in the rest frame of 2 + 3. We start from

$$R_3(s) = \int \frac{d^3\bar{p}_1}{2E_1}\frac{d^3\bar{p}_2}{2E_2}\frac{d^3\bar{p}_3}{2E_3}\delta^4(p_a + p_b - p_1 - p_2 - p_3)$$

$$= \int ds_2 \left\{ \int \frac{d^3\bar{p}_1}{2E_1}\frac{d^3\bar{p}_{23}}{2E_{23}}\delta^4(p_a + p_b - p_1 - p_{23}) \right\} \left\{ \int \frac{d^3\bar{p}_2}{2E_2}\frac{d^3\bar{p}_3}{2E_3}\delta^4(p_{23} - p_2 - p_3) \right\}$$

where we have used $\int ds_2 \int \frac{d^3\bar{p}_{23}}{2E_{23}}\delta^4(p_{23} - p_2 - p_3) = 1$ with $E_{23}^2 = \bar{p}_{23}^2 + s_2$

Now the two integrals (both of type R_2) inside the brackets are familiar from 2 → 2 scattering and 1 → 2 decay:

Now $\int \frac{d^3\bar{p}_2}{2E_2}\frac{d^3\bar{p}_3}{2E_3}\delta^4(p_{23} - p_2 - p_3) = \frac{P_3^{R23}}{4\sqrt{s_2}}\int d\Omega_3^{R23} = \frac{\pi\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{2s_2}$ and

$$\int \frac{d^3\bar{p}_1}{2E_1}\frac{d^3\bar{p}_{23}}{2E_{23}}\delta^4(p_a + p_b - p_1 - p_{23}) = \int \frac{P_1^* d\cos\theta_1 d\phi_1}{4\sqrt{s}} = \frac{\pi\int dt}{4P_a^*\sqrt{s}} = \frac{\pi\int dt}{2\sqrt{\lambda(s,m_a^2,m_b^2)}}$$

so finally $\qquad R_3(s) = \int dt_1 ds_2 \frac{\pi^2\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{4s_2\sqrt{\lambda(s,m_a^2,m_b^2)}}$

The physical region of the process $m_a + m_b \to m_1 + \sqrt{s_2}$ is

$$m_2 + m_3 \le \sqrt{s_2} \le \sqrt{s} - m_1 \quad \wedge \quad |\cos\theta_1^*| \le 1, \quad \Rightarrow$$

$$\lambda(s,s_2,m_1^2) \ge 0 \wedge \lambda(s_2,m_2^2,m_3^2) \ge 0 \quad \wedge \quad G(s,t_1,s_2,m_a^2,m_b^2,m_1^2) \le 0$$

(E.Byckling & K.Kajantie)

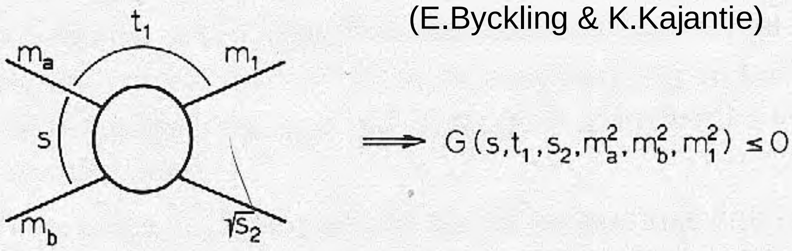


Figure V.5.2

Chew–Low plot

The physical region for the 2 → 3 process given in the $t_1 s_2$ plane (defined by the condition $G \le 0$) is called **the Chew–Low plot**. To obtain its boundary, one can solve the equation either for t_1 in terms of s_2 or vice versa.

$$t_1^{\pm} = m_a^2 + m_1^2 - \frac{1}{2s}\left\{(s + m_a^2 - m_b^2)(s - s_2 + m_1^2) - \pm\sqrt{\lambda(s, m_a^2, m_b^2)}\sqrt{\lambda(s, s_2, m_1^2)}\right\}$$

$$s_2^{\pm} = s + m_1^2 - \frac{1}{2m_a^2}\left\{(s + m_a^2 - m_b^2)(m_a^2 + m_1^2 - t_1) - \pm\sqrt{\lambda(s, m_a^2, m_b^2)}\sqrt{\lambda(t_1, m_a^2, m_1^2)}\right\}$$

These curves are 2nd order in t_1 and s_2: the boundary is a branch of a hyperbola with a lower & upper bound on s_2 of $(m_2 + m_3)^2$ & $(\sqrt{s} - m_1)^2$. An example is shown below

(E.Byckling & K.Kajantie)

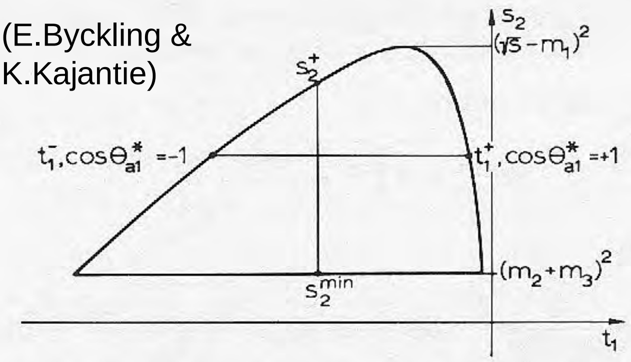


Figure V.5.3 The Chew–Low plot. The quantities $t_1^{\pm}$ and $s_2^{\pm}$ are given in equations (5.10–11). In this case the hyperbola $G = 0$ reaches the value $t_1 = (m_a - m_1)^2$ outside the physical region

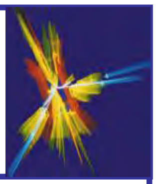
The invariants s_2 & t_1 are given in CMF by

$$s_2 = s + m_1^2 - 2\sqrt{s}E_1^*,$$

$$t_1 = m_a^2 + m_1^2 - 2E_a^* E_1^* +$$
$$2P_a^* P_1^* \cos\theta_1^*$$

The expression for t_1 shows that for a fixed s_2, t_1 depends linearly on $\cos\theta_{a1}^*$ (i.e. one moves horizontally in the Chew–Low plot by changing $\cos\theta_{a1}^*$). Substituting $t_1 = (m_a - m_1)^2$ in the formula for $s_2^{\pm}$, one obtains that $s_2^+ = s_2^-$, which shows that the line $t_1 = (m_a - m_1)^2$ is tangent to the hyper-bola $G = 0$. This is the maximum value that t_1 can attain & corresponds to equal velocity situation for particles a & 1.

Chew-Low plot



R_3 with all the conditions looks like

$$R_3(s) = \frac{1}{4\sqrt{\lambda(s,m_a^2,m_b^2)}}\int_0^{2\pi} d\phi \int dt_1 ds_2 \Theta\left\{-G(s,t_1,s_2,m_a^2,m_b^2,m_1^2)\right\} \times$$

$$\Theta\left\{\lambda(s,s_2,m_1^2)\right\}\Theta\left\{\lambda(s_2,m_2^2,m_3^2)\right\}\frac{\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{8s_2}\int d\Omega_3^{R23}$$

The phase space density in the Chew–Low plot is then

$$\frac{d^2R_3}{ds_2 dt_1} = \frac{\pi^2\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{4s_2\sqrt{\lambda(s,m_a^2,m_b^2)}}$$

the differential is constant with t_1, but varies with s_2.

One can further integrate over either s_2 or t_1. Since the integrand is independent of t_1, the integration over t_1 only gives a factor $t_1^+ - t_1^-$. The result is, of course, identical to the one obtained from the Dalitz plot. Integrating over s_2:

$$\frac{dR_3}{dt_1} = \frac{\pi^2}{4\sqrt{\lambda(s,m_a^2,m_b^2)}}\int_{s_2^{\min}}^{s_2^+} ds_2 \frac{\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{s_2},$$

$$\text{where} \quad s_2^{\min} = \max\{s_2^-,(m_2+m_3)^2\}$$

If we abbreviate $\Omega_3^{R23} = \Omega = (\cos\theta,\ \phi)$, its distribution is:

$$w(\cos\theta,\phi) \propto \int dt_1 ds_2 \frac{\sqrt{\lambda(s_2,m_2^2,m_3^2)}}{8s_2}|\mathrm{M}(s_2,t_1,\Omega)|^2,$$

where $\mathrm{M}(s_2, t_1, \Omega)$ is the matrix element for $p_a + p_b \rightarrow p_1 + p_2 + p_3$. If any variation is seen experimentally in $w(\cos\theta,\ \phi)$, it has to be due to a dependence of M in Ω.