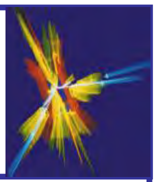


Phase space

In the reaction $a+b \rightarrow 1+...+n$, the final state is constrained by the initial through **four-momentum conservation** i.e.

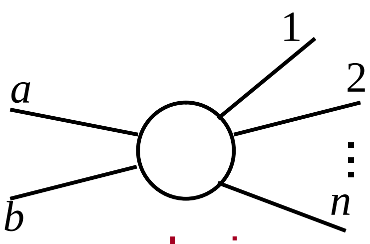
$$p_a^\mu + p_b^\mu = \sum_{i=1}^{n} p_i^\mu \quad \Leftrightarrow \quad E_a + E_b = \sum_{i=1}^{n} E_i \quad , \quad \overline{p}_a + \overline{p}_b = \sum_{i=1}^{n} \overline{p}_i$$

This is valid for ”asymptotic” states, in intermediate states the energy or momentum conversation can be violated for a very brief moment according to Heisenbergs uncertainty relation.

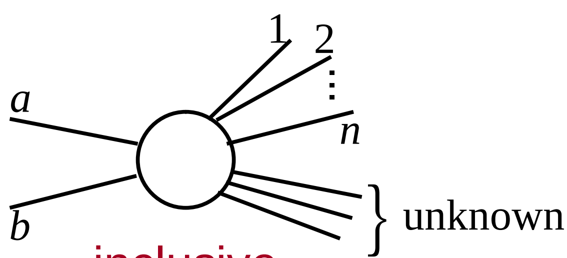
Define the 3n dimensional space of the unconstrained final state momentum vectors $\overline{p}_i$, the **momentum space.** The conditions above define in this space a 3n–4 dimensional surface, which will be called **phase space.**

To exhibit dynamical features of data or to formulate models, one needs variables, such as invariant masses or momentum transfers, in terms of which the description of the phase space often turns out to be complex. The simplification of this phase space description is one of the main aims of this course.

Need to distinguish 2 types of reactions or measurements:

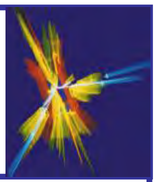


exclusive



inclusive

The reaction channel is fixed in an exclusive reaction, whereas an inclusive is a sum over several different exclusive channels.



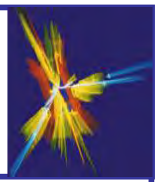
Two types of exclusive processes encountered in practice:

a particle decay, $0 \to 1 + ... + m$
a collision of particles, $a + b \to 1 + ... + n$
One can call the 1st a 1 $\to$ m & the latter a 2 $\to$ n process.
NB! When calculating essential variables for these processes, spin will be neglected throughout the course to simply things.

In e.g. a 1 $\to$ m process, there are 3m–4 free variables after the 4–momentum constraint. However, absence of spin implies that in the rest frame of the decaying particle, the orientation of the momentum configuration is irrelevant & hence 3 variables are trivial (all the "angles"). There remain 3m–7 **essential variables.**

If m = n +1, $p_a = p_0$, $p_b = -p_m$, it can be seen that particle decay and collision are related by **crossing**, i.e. particle collision can be obtained from decay by moving one of the final state particles to the initial state and vice-versa. The connection is deeper than that and e.g. invariant variables have the same physical regions.

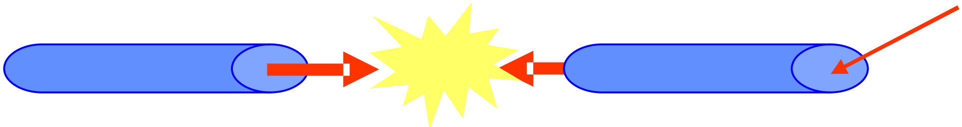
	number of variables 1 $\to$ m	example 1 $\to$ 3	number of variables 2 $\to$ n	example 2 $\to$ 2
all variables			3n–3	s, t, ϕ
essential variables			3n–4	s, t
final state variables	3m–4	$s_1, s_2, \theta_1, \phi_1, \phi$	3n–4	t, ϕ
essential final state variables	3m–7	s_1, s_2	3n–5	t



The concept of cross sections

Cross sections σ or differential cross sections $d\sigma/d\Omega$ are used to express the probability of interactions between elementary particles.

Example 2 colliding particle beams



What is the interaction rate $R_{int.}$?

$$R_{int} \propto \underbrace{f\, N_1 N_2 / A}_{\text{Luminosity } L\ [cm^{-2}\, s^{-1}]} = \sigma \cdot L$$

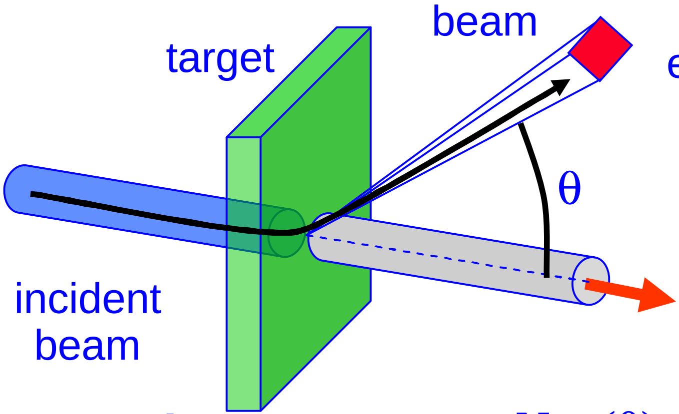
σ has dimension area !

Practical unit:

1 barn (b) = 10^{-24} cm^2

$$N_{int} = R_{int}\, t$$

Example: Scattering from target



n_A = area density of scattering centers in target

$$N_{scat}(\theta) \propto N_{inc} \cdot n_A \cdot d\Omega$$
$$= d\sigma/d\Omega\,(\theta) \cdot N_{inc} \cdot n_A \cdot d\Omega$$

Define **luminosity** precisely:

imagine a particle colliding with a bunch of cross section area – A. Probability of collision is:

(E.Wilson)

$$\sigma \cdot N_{part/bunch} / A$$

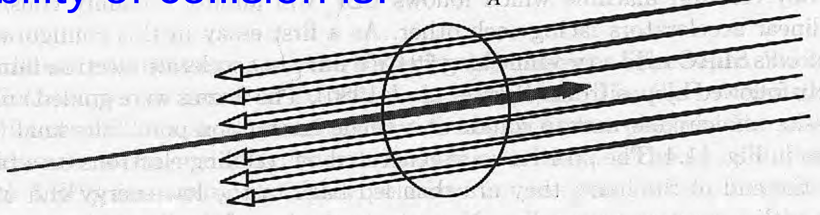


Fig. 11.3 A probe particle encounters a target—a beam of particles with cross sectional area A travelling in the opposite sense.

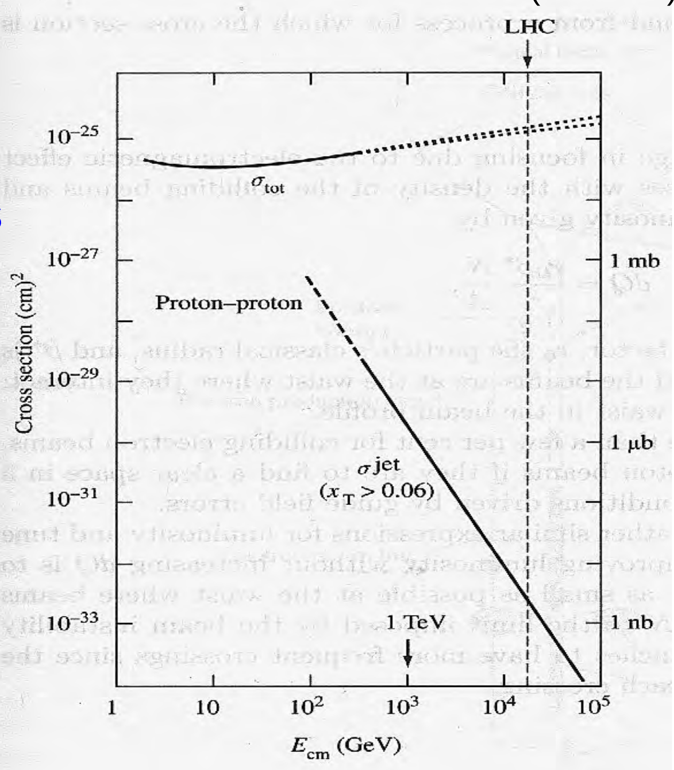
for $N_{part/bunch}$ particles in both beams $\sigma \cdot N^2_{part/bunch} / A$

and finally take into account the bunch crossing frequency f_b = # of bunches multiplied by the revolution frequency.

$$\text{Event rate} = L \cdot \sigma, \text{ where } \quad L = \frac{f_b N^2_{part/bunch}}{A}$$ **(= luminosity)**

(E.Wilson)

Ultimate challenge to high energy colliders: the production rate of "interesting" interactions fall as $1/s$ ($\propto 1/E_{CM}^2$), hence need to improve luminosity a factor 100 for each factor 10 energy increase to benefit from energy increase (distances at which structures are probed $\propto 1/\sqrt{s}$).



Cross section, partial decay width & lifetime

Formally for particle reactions, the transition probability from an initial state to a final state is defined by

$$\langle \bar{p}_1, \ldots, \bar{p}_n | \mathrm{M} | \bar{p}_a, \bar{p}_b \rangle \equiv \mathrm{M}(\bar{p}_i)$$ the "matrix element".

The matrix M will contain all the "physics" of the reaction and will not be a subject of this course. We simply note that it is an unknown function of the $\bar{p}_i$'s. To obtain measurable quantities, the square $|\mathrm{M}(\bar{p}_i)|^2$ of the matrix element has to be integrated over a set of allowed values of $\bar{p}_i$'s.

The **cross section** is obtained by integrating over the entire 3n–4 dimensional phase space for all possible values of $\bar{p}_i$. Corresponding quantity for decay is the **partial decay width**.

Cross section: $\sigma_n \equiv \sigma_n(s, m_i) = I_n(s)/F$, where

$F = 4(2\pi)^{3n-4}\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ is the flux factor &

$I_n(s) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i) |\mathrm{M}(\bar{p}_i)|^2$ contains the integration over phase space. NB! definition is a convention.

Partial decay width: $\Gamma_m = \frac{1}{2m} \frac{1}{(2\pi)^{3m-4}} I_m(m^2)$, where

$$I_m(m^2) = \int \prod_{i=1}^{m} \frac{d^3 p_i}{2E_i} \delta^4(p - \sum_{i=1}^{m} p_i) \left| \langle \bar{p}_1, \ldots, \bar{p}_m | \mathrm{M} | \bar{p} \rangle \right|^2 .$$

The **lifetime** τ of an unstable particle is the inverse of Γ_{tot}, the sum of the partial decay widths of all possible decays,

$$\frac{1}{\tau} = \Gamma_{\text{tot}} = \sum_j \Gamma_j, \quad \text{similarly} \quad \sigma_{\text{tot}} = \sum_j \sigma_j$$

Differential cross section & variable change

If the integration is restricted to a subset of the available phase space, a **differential cross section** is obtained. In practice this is most simply done by inserting δ functions.

$$\frac{d\sigma_n}{dx} = \frac{1}{F}\int\prod_{i=1}^{n}\frac{d^3p_i}{2E_i}\delta^4(p_a + p_b - \sum_{i=1}^{n} p_i)\delta(x - x(\overline{p}_i))\left|\mathrm{M}(\overline{p}_i)\right|^2$$

$x = x(\overline{p}_i)$ assumed. This trivially satisfies $\int dx\,(d\sigma_n/dx) = \sigma_n$. Higher-order differential cross sections $d^2\sigma_n/dxdy$ etc..., are obtained similarly by inserting just more δ functions.

Often one is not so interested in or not experimentally able to measure absolute numbers but only the shape of the differential distribution. Then one can define $w(x)$:

$w(x) = \frac{1}{\sigma}\frac{d\sigma}{dx}$, that is normalized to unity $\int dx\ w(x) = 1$.

Often one is faced by the situation that a prediction of a differential distribution is given in one frame and the measurement is done in another. Hence one needs to change the variables in the distribution function. That can be done very generally using **Jacobian determinants**, e.g. in the case of a three-variable function $w(x', y', z')$:

$$w'(x',y',z') = \frac{1}{\sigma}\frac{d^3\sigma}{dx'dy'dz'} = \frac{1}{\sigma}\frac{d^3\sigma}{dxdydz}\frac{\partial(x,y,z)}{\partial(x',y',z')} = \frac{\partial(x,y,z)}{\partial(x',y',z')}w(x,y,z)$$

here the Jacobian is : $\frac{\partial(x,y,z)}{\partial(x',y',z')} = \begin{vmatrix} \partial x/\partial x' & \partial y/\partial x' & \partial z/\partial x' \\ \partial x/\partial y' & \partial y/\partial y' & \partial z/\partial y' \\ \partial x/\partial z' & \partial y/\partial z' & \partial z/\partial z' \end{vmatrix}$

Example of variable change

You are all familiar to the interchange between cartesian and polar coordinates and know that in momentum space the differential $dp_1 dp_2 dp_3 = p^2 \sin\theta \; dp d\theta d\phi = p^2 dp d\cos\theta d\phi.$

Derive the above expression as an example of Jacobians.

$$p_1 = p\sin\theta\cos\phi, \quad p_2 = p\sin\theta\sin\phi, \quad p_3 = p\cos\theta \quad \Rightarrow$$

$$\frac{\partial p_1}{\partial p} = \sin\theta\cos\phi, \quad \frac{\partial p_1}{\partial \theta} = p\cos\theta\cos\phi, \quad \frac{\partial p_1}{\partial \phi} = -p\sin\theta\sin\phi, \quad \frac{\partial p_2}{\partial p} = \sin\theta\sin\phi$$

$$\frac{\partial p_2}{\partial \theta} = p\cos\theta\sin\phi, \quad \frac{\partial p_2}{\partial \phi} = p\sin\theta\cos\phi, \quad \frac{\partial p_3}{\partial p} = \cos\theta, \quad \frac{\partial p_3}{\partial \theta} = -p\sin\theta, \quad \frac{\partial p_3}{\partial \phi} = 0$$

From previous page: $dp_1 dp_2 dp_3 = \frac{\partial(p_1, p_2, p_3)}{\partial(p, \theta, \phi)} dp d\theta d\phi$

$$\frac{\partial(p_1, p_2, p_3)}{\partial(p, \theta, \phi)} = \begin{vmatrix} \frac{\partial p_1}{\partial p} & \frac{\partial p_2}{\partial p} & \frac{\partial p_3}{\partial p} \\ \frac{\partial p_1}{\partial \theta} & \frac{\partial p_2}{\partial \theta} & \frac{\partial p_3}{\partial \theta} \\ \frac{\partial p_1}{\partial \phi} & \frac{\partial p_2}{\partial \phi} & \frac{\partial p_3}{\partial \phi} \end{vmatrix} = \frac{\partial p_1}{\partial p}\cdot\frac{\partial p_2}{\partial \theta}\cdot\frac{\partial p_3}{\partial \phi} + \frac{\partial p_2}{\partial p}\cdot\frac{\partial p_3}{\partial \theta}\cdot\frac{\partial p_1}{\partial \phi} +$$

$$\frac{\partial p_3}{\partial p}\cdot\frac{\partial p_1}{\partial \theta}\cdot\frac{\partial p_2}{\partial \phi} - \frac{\partial p_1}{\partial \phi}\cdot\frac{\partial p_2}{\partial \theta}\cdot\frac{\partial p_3}{\partial p} - \frac{\partial p_2}{\partial \phi}\cdot\frac{\partial p_3}{\partial \theta}\cdot\frac{\partial p_1}{\partial p} - \frac{\partial p_3}{\partial \phi}\cdot\frac{\partial p_1}{\partial \theta}\cdot\frac{\partial p_2}{\partial p} =$$

$$p^2(0 + \sin^3\theta\sin^2\phi + \cos^2\theta\cos^2\phi\sin\theta + \sin\theta\sin^2\phi\cos^2\theta + \sin^3\theta\cos^2\phi - 0) =$$

$$p^2(\sin^3\theta + \cos^2\theta\sin\theta) = p^2\sin\theta(\sin^2\theta + \cos^2\theta) = p^2\sin\theta$$

So $dp_1 dp_2 dp_3 = p^2 \sin\theta \; dp d\theta d\phi$ as we have been thought.

Something else useful to know about Jacobians, the "chain rule":
$$\frac{\partial(x_1,\ldots,x_n)}{\partial(z_1,\ldots,z_n)}=\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\frac{\partial(y_1,\ldots,y_n)}{\partial(z_1,\ldots,z_n)}=\ldots$$

a special case of use is:
$$\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\frac{\partial(y_1,\ldots,y_n)}{\partial(x_1,\ldots,x_n)}=1$$

Let's return to the integral $I_n(s)$, which includes factors $d^3p_i/2E_i$. They are Lorentz invariant as can be seen by differentiating the 4–momentum transformation formulas.

$$dp_{x(y)}=dp'_{x(y)} \qquad dp_z=\gamma(dp'_z+vdE')=\gamma\,dp'_z(1+vp'_z/E')=dp'_zE/E'$$

since $dE'/dp'_z=p'_z/E'$ and $E=\gamma(E'+vp'_z)$. The volume element $d^3p=dp_xdp_ydp_z$ thus satisfies $d^3p/E=d^3p'/E'$ so the combination d^3p/E is invariant.
Rewritten into integral form for a timelike p:

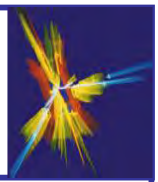
$$d^3p/2E=\int d^4p\,\delta(p^2-m^2)\Theta(p^0)$$

where $\Theta(p^0)$ is a step function that is zero for $p^0<0$ and 1 for $p^0>0$. The δ function integration has following property

$$\delta(f(x))=\delta(x-x_0)/|f'(x_0)|, \quad \text{where } f(x_0)=0$$ so

$$\int d^4p\,\delta(p^2-m^2)\Theta(p^0)=\int d^3p\,dp^0\,\delta((p^0)^2-E^2)\Theta(p^0)=\frac{d^3p}{2E}\int dp^0\,\delta(p^0-E)\Theta(p^0)$$

Now the factor 2 that is conventially added gets an explaination. Note that the Θ function is usually omitted.



So now the integral over the phase space is:

$$I_n(s) = \int \prod_{i=1}^{n} d^4 p_i \delta(p_i^2 - m^2) \Theta(p^0) \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i) \left| \mathrm{M}(\overline{p}_i) \right|^2$$

In applications, the momentum integral above usually has to be transformed into another set of variables if e.g. M is expressed in terms of some dynamically motivated variables or just involve variables that are not momentum ones. The δ function is a singular function and has to be eliminated in e.g. numerical calculations. After eliminating the δ-functions, one has 3n–4 variables only constrained by the limits of integration, defined as variables Φ

$$I_n(s) = \int d\Phi \rho_n(\Phi) \left| \mathrm{M}(\Phi) \right|^2$$

where $\rho_n(\Phi)$ is **the phase space density** containing all factors arising from transformations between the momentum variables and the Φ variables.

When M is set to 1, we define the **phase space integral**

$$R_n(s) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i)$$

This integral has no physical meaning but is technically useful since e.g. $\rho_n(\Phi)$ and the physical region of Φ are independent of the matrix element. Thus most kinematics can be done without knowing the matrix element. The next chapters will largely deal with transformations of R_n.