Smoothed particle hydrodynamics for relativistic heavy-ion collisions

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Abstract
A smoothed particle hydrodynamics (SPH) method is developed for the study of relativistic heavy-ion collisions. In order to describe the flow of a high-energy but low baryon number density fluid, entropy is taken as the SPH base. The method is formulated in terms of a variational principle, and the SPH equations for an arbitrary curvilinear coordinate system are obtained. Several examples show that SPH is a promising tool for the study of hadronic flow in relativistic heavy-ion processes.

1. Introduction

Hydrodynamic descriptions of high-energy hadronic and nuclear collisions have a rather long history [1]. Although from the theoretical point of view it is not a trivial matter to justify their validity, they have been successful in reproducing certain features of these processes, such as the energy dependence of the average multiplicity and the transverse-energy distributions. More recently, fluid dynamics has become an important tool for the analysis of relativistic heavy-ion collisions (see, for example, [2–4] and references therein). In these processes, nuclear matter is expected to be compressed and heated up close to those states of matter realized soon after the big bang. Some laboratory data on these fascinating processes have already been obtained in the series of CERN experiments [5], and data at even higher temperatures and densities are now becoming available in the relativistic heavy-ion collisional (RHIC) experiments.

Relativistic hydrodynamics is a description based on local conservation laws, together with the hypothesis of local thermodynamical equilibrium. The conservation laws are written in terms of the 4-divergence of the energy–momentum tensor. The resulting system of equations is highly nonlinear, and analytical solutions are only available for some very particular and limited configurations and equations of state [1, 6]. Thus numerical approaches are resorted to, but when no symmetry is present these methods become computationally very expensive [7]. This is exactly the case of realistic simulations of nuclear collision processes, where in general we expect no geometrical symmetry so that a full three-dimensional (3D) calculation is required.

One basic point in the hydrodynamic approach of relativistic nuclear collisions is that its main ingredients, i.e. the equation of state and the initial conditions, are not quite well known. On the contrary, we apply the hydrodynamic models to infer this information from comparisons with data. This means that we need to perform many hydrodynamical calculations for different equations of state and initial conditions. In such a process, we actually do not need a very precise solution of the hydrodynamic equations, but a general flow pattern which characterizes the final configuration of the system as a response to a given equation of state and initial condition. We are not interested in, and are probably not able to analyse at least at the present stage, any very precise local feature (sound ripples, small local perturbations, etc) in these models. Extremely local properties, therefore, should be averaged out without distorting the general flow pattern. This is particularly the case when one considers the questionable validity of local thermal equilibrium in the problem of interest here. In short, for the study of hydrodynamic models of relativistic nuclear collisions, we prefer a rather simple scheme of solving the hydrodynamic equations, not unnecessarily too precise but robust enough to deal with any kind of geometry. From this point of view, we stressed in [8] the advantage of a variational approach to relativistic hydrodynamics.

Among many numerical approaches, smoothed particle hydrodynamics (SPH) [9, 10] fits perfectly in with the variational formalism. Consequently, this method presents many desirable properties of the variational approach, such as simplicity and robustness with respect to changes in geometry, as well as the possibility of smoothing out undesirable local degrees of freedom. Furthermore, SPH parametrizes the matter flow in terms of discrete Lagrangian coordinates (called 'particles') attached to some conserved quantity, such as baryon number. In this aspect, the SPH method is well suited for studies of relativistic nuclear collisions, where an extremely compressed and high-temperature hadronic system expands into a very large space region. The Lagrangian nature is one of the main advantages of SPH over the Eulerian space-fixed grid algorithms when applied to the study of RHIC physics.

The SPH algorithm was first introduced for astrophysical applications and has been used in calculations of processes such as collisions of neutron stars, supernova explosions, fragmentation of asteroids and many others. Several extensions of the SPH method to the ultrarelativistic regime of hydrodynamics have been made [11–14]. However, some specific aspects of relativistic heavy-ion processes deserve attention when applying SPH. One of them is that, in the ultrarelativistic regime of central collisions, a large fraction of the incident energy is converted into produced particles. In particular, in the mid-rapidity region of central collisions most of this energy is in the form of produced mesons and only a very small portion is carried by baryons. This is a very unfavourable situation to the conventional formulation of the SPH algorithm, where the computational particles are associated with the baryon number density. A direct application of an SPH method based on the conserved baryon number may fail in the baryon-poor, meson-dominated region.

As mentioned above, the basic point of the SPH method is to introduce a set of 'particles' which follow the flow of the fluid. However, the definition of flow does not necessarily rely on a conserved quantity. For example, according to Landau [19] flow is defined in terms of the local Lorentz frame in which the energy–momentum tensor becomes diagonal. In this paper, we explore this aspect and formulate SPH in terms of any extensive quantities defined in the Landau comoving local frame. We derive the relativistic SPH equations using a variational principle [8], taking matter flow as the variable. We argue that the convenient quantity to attribute to SPH particles in the case of relativistic heavy-ion collisions is the entropy of the fluid. In this way, we can follow directly the entropy content and its change due to the dissipative processes such as shock waves. This is particularly interesting for systems where a first-order phase transition is present [15–17].

Another specific aspect of relativistic heavy-ion collisions is how to set the initial conditions for the hydrodynamic motion. As mentioned before, this is not a solved problem. However, generally speaking, the hydrodynamic regime will be established from the initial non-equilibrium state of microscopic quantum chromodynamics (QCD) degrees of freedom only after a certain relaxation time τ_{relax}. This means that the hypersurface of the onset of the hydrodynamic regime is characterized by a nearly constant local proper time rather than by a constant global coordinate time t. For the ultrarelativistic regime this difference becomes crucial. In the limit of very large initial energy, the Bjorken scaling solution is expected to be a good first-order approximation at least for the longitudinal motion. Therefore, we shall write the hydrodynamic equations in terms of the Bjorken scaling coordinates $\tau = \sqrt{t^2 - z^2}$ and $\eta = \frac{1}{2}\ln(t+z)/(t-z)$, replacing t and z, as far as the longitudinal motion is concerned. These variables are also suitable for the set-up of the initial conditions. Here, we show that the variational approach is also useful for the derivation of the SPH equations in an arbitrary curvilinear coordinate system.

We organize the present paper as follows. In section 2 we introduce the relativistic variational formulation with the SPH parametrization of one of the thermodynamical extensive quantities. In particular, we argue that entropy is a convenient extensive quantity for applications to relativistic heavy-ion collisions. In this case, the change of entropy due to dissipative processes should be taken into account. This is particularly important in the presence of a shock wave or a first-order phase transition. For this purpose, we deduce the entropy-based SPH equations in the presence of bulk viscosity. In section 3 we apply our formulation to several relativistic systems such as the Landau model (in one and three dimensions), and relativistic shock tubes. These results are compared with known solutions. Section 4 is dedicated to the conclusions and perspectives of this paper.

2. Variational derivation of the relativistic SPH equation

2.1. SPH Representation of extensive variables

Non-relativistic SPH can be formulated in terms of a variational principle. We show here that the relativistic SPH can also be formulated in this way [8]. This guarantees that the SPH coordinates $\{\vec{r}_i(t)\}$ are the optimal dynamical parameters to minimize the SPH-model action. The other advantage of this approach is that it leads to a form of the SPH equations that automatically conserves linear and angular momenta [18]. This is the natural consequence of the Lorentz scalar nature of the action.

In relativistic hydrodynamics, we assume that at any spacetime point $x = \{\vec{r}, t\}$ there exists a local reference frame where the energy–momentum tensor becomes diagonal and takes the form [19]

$$T^{\mu\nu}(\vec{r}, t) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{1}$$

where ε and P are the (proper) energy density and pressure of the fluid. From the hypothesis of local equilibrium, we assume that the thermodynamical relations are valid in each local frame.

Let A be an arbitrary thermodynamical extensive quantity of the fluid such as the baryon number or entropy. The amount of A contained in the infinitesimal volume element $\mathrm{d}V$ is

denoted by dA so that the corresponding (proper) density a is

$$a = \frac{dA}{dV}.$$ (2)

This is related to the corresponding density a^* measured in the space-fixed (calculational) frame as

$$a^*(\vec{r}, t) = \gamma a$$ (3)

where γ is the local Lorentz factor associated with the flow.

In the SPH representation, we parametrize this density by the following ansatz:

$$a^*(\vec{r}, t) = \sum_i \nu_i W(\vec{r} - \vec{r}_i(t); h)$$ (4)

where $W(\vec{r} - \vec{r}'; h)$ is a positive-definite kernel function peaked at $\vec{r} = \vec{r}'$ with the normalization

$$\int d^3\vec{r} \, W\left(\vec{r} - \vec{r}'; h\right) = 1.$$ (5)

The parameter h represents the width of the kernel. In the limit, $h \rightarrow 0$, we have

$$\lim_{h \rightarrow 0} W(\vec{r} - \vec{r}'; h) = \delta^3(\vec{r} - \vec{r}').$$

As will be seen later, it is convenient to choose an even function for W. It can be a Gaussian function in $x = |\vec{r} - \vec{r}'|/h$, but in practice we often use B-spline functions [18]. The role of W with a finite value of h is to introduce a sort of short-wavelength cut in the Fourier representation of the density a^*. The total amount of A of the system is obtained by integrating equation (4) over the whole space,

$$A_{\text{tot}} = \int d^3\vec{r} \, a^*(\vec{r}, t) = \sum_i \nu_i.$$ (6)

Physically speaking, it is clear from the above expression that we are replacing the continuous fluid by a collection of 'SPH particles' each of which carries a portion ν_i of the extensive quantity A.

The velocity of these particles are identified as the velocity of the fluid at their position $\vec{r}_i(t)$,

$$\vec{v}_i = \frac{d\vec{r}_i}{dt}$$ (7)

so that the Lorentz factor of the ith particle is given by $\gamma_i = 1/\sqrt{1 - v_i^2}$. With this identification, if $\{\nu_i\}$ are constant in time then A is a conserved quantity, as seen from the continuity equation

$$\frac{\partial a^*}{\partial t} = -\sum_i \nu_i \vec{v}_i \cdot \nabla W(\vec{r} - \vec{r}_i(t); h)$$

$$= -\nabla \cdot \sum_i \nu_i \vec{v}_i W(\vec{r} - \vec{r}_i(t); h)$$

$$= -\nabla \cdot \vec{j}_A$$

where

$$\vec{j}_A = \sum_i \nu_i \vec{v}_i W(\vec{r} - \vec{r}_i(t); h)$$

is the SPH representation of the current density of A. If $\{v_i\}$ are not constant in time then A is not conserved. In this case the continuity equation has a contribution from the time derivative of the v_i as

$$\frac{\partial a^*}{\partial t} + \nabla \cdot \vec{j}_A = \sum_i \dot{v}_i W \left(\vec{r} - \vec{r}_i(t); h\right). \tag{8}$$

We consider the set of time-dependent variables $\{\vec{r}_i, i = 1, \ldots, n\}$ as the variational degrees of freedom and their equations of motion are determined by minimizing the action for the hydrodynamic system. Here, $\{v_i\}$ are not dynamical variables and are determined by the initial conditions together with the constraints for the variational procedure (see the later discussion in section 3.3).

It may seem that the spatial extension of the particle i is identified with that of the kernel $W(\vec{r} - \vec{r}_i; h)$. However, the density of A at the position of the particle i is given by

$$a_i^* = \sum_j v_j W(\vec{r}_i - \vec{r}_j) \tag{9}$$

from equation (4). Therefore, we may define the 'specific volume' V_i of the extensive quantity A associated with the particle i as

$$V_i \equiv \frac{v_i}{a_i} = \frac{\gamma_i v_i}{\sum_j v_j W(\vec{r}_i - \vec{r}_j)}. \tag{10}$$

Any other extensive quantities carried by the particle i can be calculated easily. Let o^* be the density of another extensive quantity, say O, measured in the calculational frame. Then the amount of O carried by the particle i is

$$v_i(O/A)_i = v_i(o/a)_i = v_i(o/a)_i^* \tag{11}$$

so that the density distribution of O in the calculational frame is expressed as

$$o^*(\vec{r}, t) \to \sum_i v_i(o/a)_i^* W(\vec{r} - \vec{r}_i(t)). \tag{12}$$

2.2. SPH action

The relativistic hydrodynamic equations can be obtained by the variational principle for the action [8]

$$I = -\int d^4x \, \varepsilon \tag{13}$$

with respect to the matter density distribution, subjected to the constraint expressed by the continuity equation. As before, ε is the proper energy density of the fluid. Here, for the sake of simplicity, we discuss first the case of the Minkowski metric. The argument can readily be generalized for more general coordinate systems (see section 3). The Lagrangian of the system is

$$L = -\int d^3\vec{r} \, \varepsilon. \tag{14}$$

We may consider ε as the Lagrangian density in the space-fixed frame. Therefore, the corresponding Lagrangian for the system of SPH particles can be taken to be

$$L_{\text{SPH}} \left(\left\{ \vec{r}_i, \frac{d\vec{r}_i}{dt} \right\} \right) = -\sum_i v_i \left(\varepsilon/a^* \right)_i$$

$$= -\sum_i \left(\frac{E}{\gamma} \right)_i \tag{15}$$

where $\{\vec{r}_i(t)\}$ are the dynamical variables and $E_i = \nu_i \, (\varepsilon/a)_i$ is the 'rest energy' of the particle i. The SPH model action is then

$$I_{\text{SPH}} = -\int \mathrm{d}t \sum_i \left(\frac{E}{\gamma}\right)_i. \tag{16}$$

2.3. Variational procedure

The Lagrangian, given by equation (15), seems to show that the system behaves as if it is just a collection of free particles of 'rest mass' E_i. However, there exists a basic difference. Namely, E_i are not constant with respect to the variation of positions. This is because the energy contained in each particle is determined by the configuration of other particles through thermodynamical relations. More explicitly, variations in the particle positions $\{\vec{r}_i, i = 1, \ldots, n\}$ cause a change of the volume occupied by each particle, which in turn modifies its 'rest' energy.

For our variational procedure, we consider changes of quantities associated only kinematically with virtual variations of the configuration. That is, except for the energy and the volume, all extensive quantities such as the entropy and the particle number should be kept constant while variations of $\{\vec{r}_i\}$ are taken. This implies that for variations of $\vec{r}_i$, we keep ν_i constant if A is not the energy or the volume. Now, if we take A as the volume, then its changes associated with variations of $\vec{r}_i$ are described by equation (4) where ν_i should be understood as the initial volume of the particle i. Therefore, even for the case where A is the volume, ν_i should be kept constant for variations of $\vec{r}_i$. In short, except for the case of the energy, the parameters ν_i should be kept constant while variations of $\{\vec{r}_i\}$ are taken. If we take A as the energy, then the change of ν_i should be calculated as a function of the variation in $\{\vec{r}_i\}$. This introduces an additional complication without any practical merit. Therefore, for the sake of simplicity, we restrict ourselves to the case where A is not the energy but can be any other extensive quantity.

We can write the change of energy associated with a virtual change of volume δV as

$$\delta E = -P \, \delta V + \delta W \tag{17}$$

$$= -P_{\text{eff}} \delta V \tag{18}$$

where δW is additional work to change the volume irreversibly and P_{eff} is the effective pressure. If the change of volume is performed in a quasi-static way and there exists no dissipative force, then this effective pressure coincides with the usual pressure P. However, if there exists some irreversible process associated with the volume change, then $P_{\text{eff}} \neq P$ and we may write

$$P_{\text{eff}} = P + Q \tag{19}$$

where $Q\delta V$ is the energy change due to the irreversible process. For a quasi-static adiabatic process, $Q = 0$. The introduction of Q is important when we deal with shock phenomena (see section 3).

The variation of the volume δV for constant ν_i is calculated from equation (10) as

$$\delta V_i = V_i \left(\frac{\delta \gamma_i}{\gamma_i} - \frac{\delta a_i^*}{a_i^*}\right)$$

$$= -\frac{\nu_i}{a_i}\left(-\gamma_i^2 \vec{v}_i \cdot \delta\vec{v}_i + \frac{1}{a_i^*}\sum_j \nu_j \left(\delta\vec{r}_i - \delta\vec{r}_j\right) \cdot \nabla W_{ij}\right) \tag{20}$$

where $W_{ij} \equiv W(\vec{r}_i - \vec{r}_j; h)$. Using these relations, the variation of the action (16) with respect to $\{\vec{r}_i, i = 1, \ldots, n\}$ becomes

$$\delta I_{\text{SPH}} = -\int dt \sum_i \frac{1}{\gamma_i} \left(-(P + Q)_i \delta V_i - E_i \frac{\delta \gamma_i}{\gamma_i} \right)$$

$$= -\int dt \sum_i \delta \vec{r}_i \cdot \left\{ \frac{d}{dt} \left[\nu_i \left(\frac{\varepsilon + P + Q}{a} \right)_i \gamma_i \vec{v}_i \right] \right.$$

$$\left. + \nu_i \sum_j \nu_j \left[\frac{1}{\gamma_i^2} \left(\frac{P + Q}{a^2} \right)_i + \frac{1}{\gamma_j^2} \left(\frac{P + Q}{a^2} \right)_j \right] \nabla_i W_{ij} \right\}.$$

Here, we used the symmetry of W and the property

$$\nabla_i W_{ij} = -\nabla_j W_{ij}. \tag{21}$$

The requirement $\delta I_{\text{SPH}} = 0$ for all $\delta \vec{r}_i$ leads to

$$\frac{d}{dt} \left[\nu_i \left(\frac{P + Q + \varepsilon}{a} \right)_i \gamma_i \vec{v}_i \right] = -\sum_j \left[\frac{\nu_i \nu_j}{\gamma_i^2} \left(\frac{P + Q}{a^2} \right)_i + \frac{\nu_i \nu_j}{\gamma_j^2} \left(\frac{P + Q}{a^2} \right)_j \right] \nabla_i W_{ij}. \tag{22}$$

The corresponding hydrodynamic equation in the continuum limit [8] is

$$\partial_\mu T^{\mu\nu} = \partial_\mu \Sigma^{\mu\nu} \tag{23}$$

where

$$\Sigma^{\mu\nu} = Q \left[u^\mu u^\nu - g^{\mu\nu} \right] \tag{24}$$

is the stress tensor for the bulk viscosity Q. This equation is the same as those discussed in [12–14].

Equation (22) should be complemented by an equation which expresses the conservation of energy (and other thermodynamical quantities, if any diffusion process is present). The energy conservation is written as

$$\frac{dE_i}{dt} = -(P + Q)_i \frac{dV_i}{dt}. \tag{25}$$

On the other hand, from the second law of thermodynamics, we should have

$$\frac{dE_i}{dt} = -P \frac{dV_i}{dt} + T_i \frac{dS_i}{dt} + \mu_i \frac{dN_i}{dt} \tag{26}$$

in equilibrium, where T, S, μ, N are the temperature, entropy, chemical potential and baryon number, respectively. Thus, we obtain

$$T_i \frac{dS_i}{dt} + \mu_i \frac{dN_i}{dt} = -Q_i \frac{dV_i}{dt}. \tag{27}$$

In the absence of particle diffusion, chemical equilibrium requires

$$\mu \frac{dN}{dt} = 0 \tag{28}$$

and, in this case,

$$T_i \frac{dS_i}{dt} = -Q_i \frac{dV_i}{dt}. \tag{29}$$

2.4. Entropy representation of SPH equations

As we have mentioned in the introduction, in applications to ultrarelativistic nuclear collisions the baryon number is not a suitable quantity to represent the hydrodynamic flow, since most of the energy content is in the form of non-baryonic matter. This is particularly so in the central rapidity region. We may consider the energy content itself as the SPH base. However, as mentioned before, this choice introduces an additional constraint between the coordinates $\{\vec{r}_i\}$ and the extensive parameters $\{v_i\}$ of SPH particles, due to energy conservation, and is not desirable from a practical point of view. Therefore, we propose to take the entropy as the suitable extensive quantity for the SPH representation.

Let $s^*(t, \vec{r})$ be the entropy density in the space-fixed (calculational) frame. From equation (9), we have

$$
s_i^* = s^*(t, \vec{r}_i(t))
$$
$$
= \sum_j v_j W(\vec{r}_i - \vec{r}_j)
$$

and the equations of motion for $\{\vec{r}_i\}$ are given by equation (22), which we write in the form,

$$
\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = \vec{v}_i \tag{30}
$$

$$
\frac{\mathrm{d}\vec{\pi}_i}{\mathrm{d}t} = -\sum_j \left[\frac{v_i v_j}{s_i^{*2}}(P+Q)_i + \frac{v_i v_j}{s_j^{*2}}(P+Q)_j \right] \nabla_i W_{ij} \tag{31}
$$

where

$$
\vec{\pi}_i = v_i \left(\frac{P+Q+\varepsilon}{s} \right)_i \gamma_i \vec{v}_i \tag{32}
$$

is the momentum associated with particle i.

For the variational procedure, v_i are kept constant. This does not mean that they are constant in time. When there exists some non-adiabatic process, then $Q \neq 0$ and the energy conservation equation, equation (29), gives

$$
\frac{1}{v_i} \frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{Q_i}{T_i s_i^*} \theta_i \tag{33}
$$

where

$$
\theta_i = \frac{1}{V_i} \frac{\mathrm{d}V_i}{\mathrm{d}t}
$$
$$
= (\partial_\mu u^\mu)_i. \tag{34}
$$

Equations (30), (31) and (33) constitute a system of first-order differential equations for $\vec{r}_i$, $\vec{\pi}_i$ and v_i, where the velocity $\vec{v}_i$ should be determined algebraically from equation (32).

We also have to specify the equation of state,

$$
P = P(n, s)
$$

and the dissipative pressure,

$$
Q = Q(n, s, \theta).
$$

In general, these quantities will depend on the (proper) baryon number density n, which in the computational frame has the SPH representation

$$
n_i^* = \sum_j N_j W(\vec{r}_i - \vec{r}_j) \tag{35}
$$

where N_i is the conserved baryon number carried by particle i. Note that as we have based the SPH dynamics on entropy, there is no problem in having $N_i = 0$ for some of the particles.

2.5. SPH equation for a generalized coordinate system

The variational procedure can readily be extended to coordinate systems with a non-Cartesian metric. The use of the generalized coordinate system is particularly important when we consider realistic initial conditions for simulations of RHIC processes. In a relativistic heavy-ion collisional process, the initial state corresponds to two cold, quantum nuclear systems flying separately. Just after the collision, the hadronic matter stays in a highly excited state and the materialization occurs only after ~ 1 fm c^{-1} in the proper time. Therefore, the local thermodynamical state would emerge for some local proper time and not for the global space-fixed time t. Thus, it is important to choose a convenient coordinate system for the description of relativistic heavy-ion collisions. For example, one often uses the hyperbolic time and longitudinal coordinates to be described later.

Let us consider a more general coordinate system,

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{36}$$

However, in order to unambiguously define the conserved quantities, we consider only the case when the timelike coordinate is orthogonal to the spacelike coordinates,

$$g_{\mu 0} = 0. \tag{37}$$

The action principle for the relativistic fluid motion can be written as [8]

$$\delta I = -\delta \int \mathrm{d}^4 x \sqrt{-g}\,\varepsilon = 0 \tag{38}$$

together with the constraint for the conserved entropy current,

$$(su^\mu)_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}su^\mu\right) = 0 \tag{39}$$

or

$$\frac{1}{\sqrt{-g}}\partial_\tau\left(\sqrt{-g}s\gamma\right) + \frac{1}{\sqrt{-g}}\sum_i \partial_i\left(\sqrt{-g}s\gamma v^i\right) = 0 \tag{40}$$

where

$$v^i = \frac{u^i}{u^0} \tag{41}$$

and we use the notation,

$$\tau = x^0 \qquad \gamma = u^0.$$

The generalized gamma factor γ is related to the velocity $\vec{v}_a$ through $u_\mu u^\mu = 1$, so that

$$\gamma = \frac{1}{\sqrt{g_{00} - \vec{v}^T \boldsymbol{g}\vec{v}}} \tag{42}$$

where $-\boldsymbol{g}$ is the 3×3 space part of the metric tensor. That is

$$\left(g_{\mu\nu}\right) = \begin{pmatrix} g_{00} & 0 \\ 0 & -\boldsymbol{g} \end{pmatrix}. \tag{43}$$

Let us now introduce the SPH representation. We may, for example, express the entropy density by the ansatz

$$\sqrt{-g}\,s\gamma = s^* \quad \rightarrow \quad s^*_{\mathrm{SPH}} = \sum_i \nu_i W(\vec{r} - \vec{r}_i(\tau)) \tag{44}$$

or by

$$s\gamma = s^* \quad \rightarrow \quad s^*_{\text{SPH}} = \sum_i \nu_i W(\vec{r} - \vec{r}_i(\tau)) \tag{45}$$

as well. These two possibilities, besides others, are simply different ways to parametrize a variational ansatz in terms of a linear combination of given functions $W(\vec{r} - \vec{r}_i(\tau))$. The most important property of an ansatz should be that W satisfies the normalization condition imposed by the basic conserved quantity. Since the total entropy is expressed as

$$S = \int d^3\vec{r} \sqrt{-g}\, s\gamma = \sum_i \nu_i \tag{46}$$

the normalization of W should be taken to be

$$\int d^3\vec{r}\, W(\vec{r} - \vec{r}') = 1 \tag{47}$$

for the parametrization equation (44) and

$$\int d^3\vec{r} \sqrt{-g}\, W(\vec{r} - \vec{r}') = 1 \tag{48}$$

for the parametrization equation (45). In the usual SPH calculations, it is not desirable to introduce in W the spacetime dependence through its normalization condition. In this respect, the most natural way to introduce the SPH representation is equation (44). With this choice, the SPH action is given by

$$I_{\text{SPH}} = -\int d\tau \int d^3\vec{x} \sum_i \nu_i \left(\frac{\sqrt{-g}\varepsilon}{\sqrt{-g}s\gamma} \right)_i W(\vec{r} - \vec{r}_i(\tau))$$

$$= -\int d\tau \sum_i \nu_i \left(\frac{\varepsilon}{s\gamma} \right)_i. \tag{49}$$

The variational principle leads to the following equation of motion:

$$\frac{d}{d\tau}\vec{\pi}_i = -\sum_j \nu_i \nu_j \left[\frac{1}{\sqrt{-g_i}\gamma_i^2} \frac{P_i + Q_i}{s_i^2} + \frac{1}{\sqrt{-g_j}\gamma_j^2} \frac{P_j + Q_j}{s_j^2} \right] \nabla_i W_{ij}$$

$$+ \frac{\nu_i}{\gamma_i} \frac{P_i + Q_i}{s_i} \left(\frac{1}{\sqrt{-g}} \nabla \sqrt{-g} \right)_i$$

$$+ \frac{\nu_i}{2}\gamma_i \left(\frac{P + Q + \varepsilon}{s} \right)_i \left(\nabla g_{00} - \vec{v}_i^T \nabla g \vec{v}_i \right)_i \tag{50}$$

where

$$\vec{\pi}_i = \gamma_i \nu_i \left(\frac{P + Q + \varepsilon}{s} \right)_i g\vec{v}_i \tag{51}$$

and the operator ∇ is just the simple derivative operator with respect to the coordinate variable in use.

2.6. Hyperbolic coordinates

For ultrarelativistic heavy-ion collisions, a useful set of variables is

$$\tau = \sqrt{t^2 - z^2} \tag{52}$$

$$\eta = \frac{1}{2} \tanh \frac{t+z}{t-z} \tag{53}$$

$$\vec{r}_T = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{54}$$

As mentioned above, the initial conditions for RHIC processes are specified in terms of proper the time rather than of the coordinate time t. The variable τ is not exactly the physical proper time of the matter, but for the initial times it may approximate the proper time.

The metric tensor for this coordinate system is given by

$$g_{00} = 1$$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau^2 \end{pmatrix}$$

$$\sqrt{-g} = \tau.$$

Since the metric is space independent, we can use the parametrization,

$$\tau \gamma_i s_i = s_i^* = \sum_{j=1}^{n} \nu_j W(q_{ij})$$

where

$$q_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \tau^2 (\eta_i - \eta_j)^2}$$

and W is normalized as

$$4\pi \int_0^\infty q^2 \, dq \, W(q) = 1.$$

The SPH equation becomes

$$\frac{d}{d\tau} \vec{\pi}_i = -\frac{1}{\tau} \sum_j \nu_i \nu_j \left[\frac{1}{\gamma_i^2} \frac{P_i + Q_i}{s_i^2} + \frac{1}{\gamma_j^2} \frac{P_j + Q_j}{s_j^2} \right] \nabla_i W_{ij}$$

where the η component of the momentum is related to the velocity $v_\eta = d\eta/d\tau$ as

$$\pi_\eta = \tau^2 \nu \gamma \left(\frac{P + Q + \varepsilon}{s} \right) v_\eta$$

whereas in the transverse directions we have, with $\vec{v}_T = d\vec{r}_T/d\tau$

$$\vec{\pi}_T = \nu \gamma \left(\frac{P + Q + \varepsilon}{s} \right) \vec{v}_T.$$

The Lorentz factor is given by

$$\gamma = \frac{1}{\sqrt{1 - v_T^2 - \tau^2 v_\eta^2}}.$$

3. Examples

We have formulated a relativistic SPH method appropriate for the study of RHIC processes. In order to check its validity and efficiency, in this section we apply our method to several problems which have known analytic or numerical solutions.

3.1. Adiabatic flow of massless pion gas

Entropy is conserved in the adiabatic flow of a perfect gas. Thus, in the following examples we take $\dot{v}_i = 0$ and $Q = 0$.

3.1.1. Landau model.
Let us first investigate a well known analytically solvable flow. The Landau model is one of the few examples of relativistic fluid flow for which an analytical solution is available. Consider a one-dimensional (1D), relativistic, massless, baryon-free gas initially at rest. The equation of state is

$$P = \tfrac{1}{3}\varepsilon = C s^{4/3} \tag{55}$$

where

$$C = \left(\frac{15}{128\pi^2}\right)^{1/3}.$$

Since we have $Q = 0$ the relation between the momentum and velocity is

$$\pi = C v s^{*1/3} \gamma^{2/3} v. \tag{56}$$

In this case, v can be solved analytically in terms of π.

In figure 1(*a*) we show the entropy distribution $s^*(x, t)$ obtained with a SPH calculation in $x - t$ coordinates, together with the exact solution [1]. In this example, we took only 200 particles with $h = 0.1$ fm. As we see, in spite of a rather small number of particles, the SPH solution is quite satisfactory. Figure 1(*b*) shows the temperature distribution calculated with SPH using $\eta - \tau$ coordinates. Only 100 particles were used in this calculation, and an excellent agreement with the analytical solution was obtained.
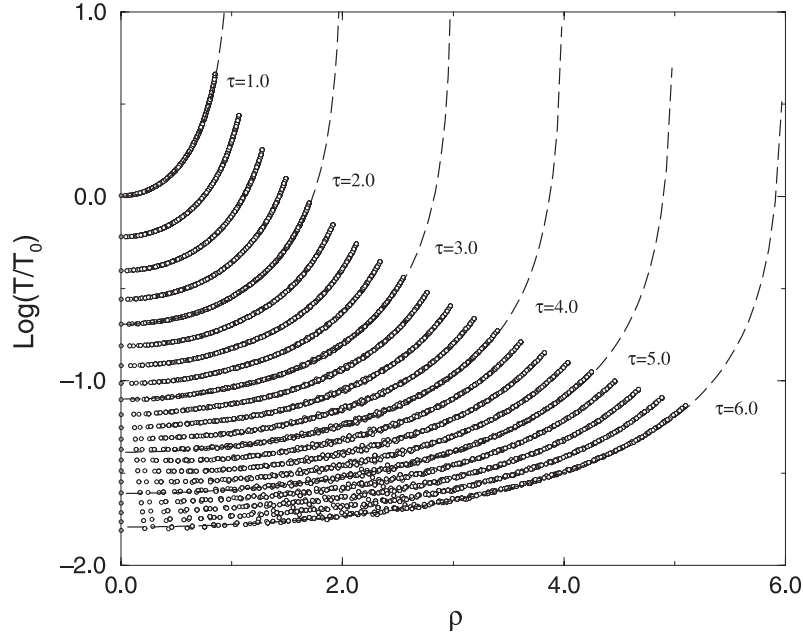
3.1.2. 3D scaling solution.
A simple analytical solution for a three-dimensional relativistic pion gas is available. It is just a generalization of the one-dimensional scaling solution and is given by

$$s = \frac{s_0}{\sqrt{\tau^2 - x^2 - y^2}}.$$

To see the efficiency of the SPH approach presented here, we reproduce this solution numerically in the full 3D numerical code, without making use of the spherical symmetry. In figure 2, we show the result of such calculations. As we see, our numerical calculation reproduces the analytical solution fairly well. One of the advantages of the SPH approach is that the coding for 3D cases is almost the same as for the 1D case. The only problem is a rapid increases of the number of particles for higher dimensions if we want to keep a high degree of accuracy. A direct coding would require a computational time proportional to n^2, where n is the number of particles. However, in the absence of long-range forces such as the gravitational or Coulomb interactions, we may apply techniques such as the linked-list method to reduce the computational time to the order of $n \log n$. In the example shown above, we used $50 \times 50 \times 50$ particles, which took less than 2 min per time step on a reasonable PC (Pentium-II). This is a rather large number of particles, since we have a somewhat stringent test due to the divergent nature of the solution at the border.
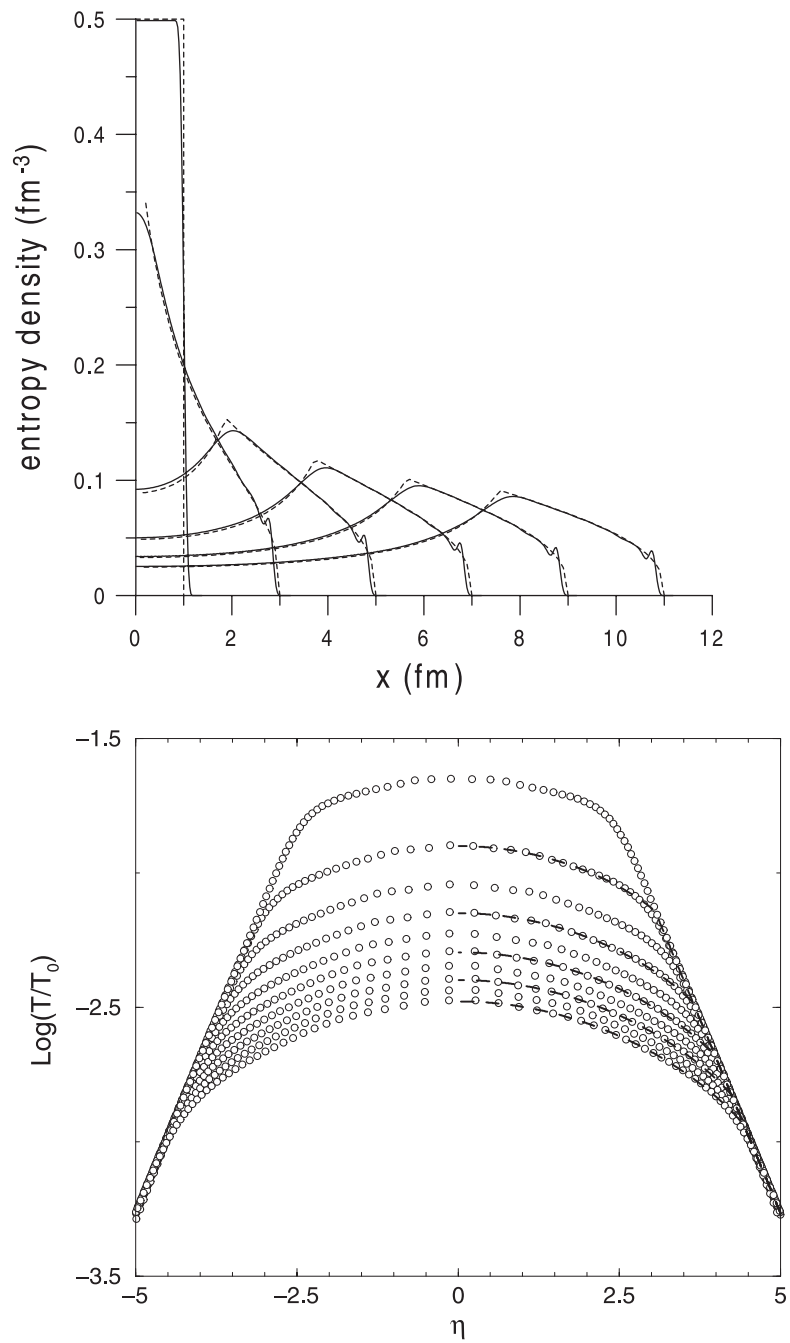
Figure 1. (*a*) Entropy profiles of the Landau model for different times. The exact results are given by the broken curves. The SPH solution in Cartesian coordinates is shown by the full curves. (*b*) Temperature profiles of the Landau model in the hyperbolic coordinate system (see text), for different times τ. The SPH calculation is represented by the circles, and the exact result by the broken curves.

Figure 2. Three-dimensional scaling solution for the massless pion gas. Cartesian coordinates are used for the transverse direction and the hyperbolic coordinates $(\eta - \tau)$ for the longitudinal direction. We show the temperature in the $\eta = 0$ plane as a function of $\rho = \sqrt{x^2 + y^2}$, for different values of τ. SPH results are represented by the circles, and the exact solution by the broken curves. Despite the spherical symmetry, the SPH calculation has been carried out in full 3D.

3.1.3. Transverse expansion on longitudinal scaling expansion. As a further test, closer to a realistic situation than that of figure 2, we calculated the transverse expansion of a cylindrically symmetric homogeneous massless pion gas, undergoing a longitudinal scaling expansion, and initially at rest in transverse directions. Such a problem has been discussed by several authors as a useful base to understand the transverse expansion. In figure 3, we compare our results (again a full 3D calculation without assuming cylindrical symmetry) with (2 + 1) numerical results, obtained using the method of characteristics [20]. In this example, we also used $50 \times 50 \times 50$ particles. The result is quite satisfactory. If we decrease the accuracy to 10%, we can reduce the particle number almost by one order of magnitude.

3.2. Non-adiabatic case: shock waves and artificial viscosity.

As seen in the previous examples, our entropy-based relativistic SPH method works quite well for the adiabatic dynamics of the massless pion gas. However, for the application to realistic problems, it is fundamental to see how this scheme works for non-adiabatic cases. For this purpose, we study some examples of one-dimensional shock problems.

3.2.1. Compression shock. Whenever there exists a shock wave, entropy is produced through the shock front. The shock front manifests as a discontinuity in thermodynamical quantities in a hydrodynamic solution. Mathematically speaking, the shock front should be treated as a boundary connecting two distinct hydrodynamic solutions. To reproduce such a discontinuous behaviour, the full degrees of freedom of hydrodynamics are required.
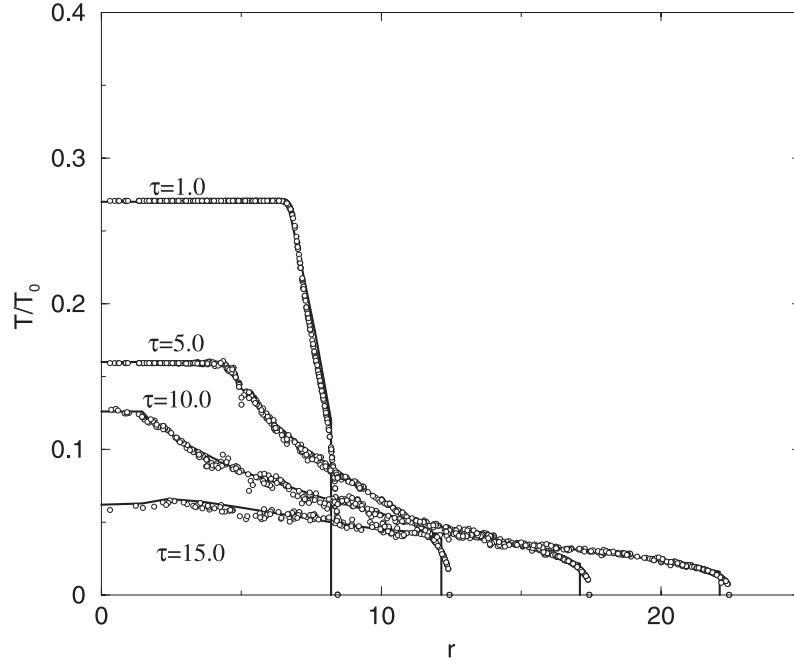
Figure 3. Temperature profiles of a cylindrically symmetric flow with longitudinally scaling expansion, shown as a function of $r = \sqrt{x^2 + y^2}$. The SPH results at $\eta = 0$ (circles) are compared with the numerical solution obtained by a space-fixed grid method. The SPH calculation has been performed in full 3D.

The smoothed particle ansatz excludes such a possibility from the beginning. Since short-wavelength excitation modes do not exist in the SPH ansatz, the energy and momentum conservation required by hydrodynamics results in a very rapidly oscillating motion of each particle. Such a situation occurs, for example, when a high-energy density gas is released into a low-density region. This kind of shock, for the case of a baryon gas, is discussed in [21] and also, in the SPH context, in [13]. Here, we apply our entropy-based SPH approach to shocks in the massless pion gas. Figure 4 gives the typical behaviour of the SPH solution for such a situation, if entropy production is not taken into account ($Q = 0$). The calculation was performed with 1000 particles and $h = 0.5$ fm. As discussed above, there appear rapid oscillations in thermodynamical quantities just behind the shock front. Such oscillations always appear in numerical approaches if entropy production is not included.

In order to avoid these oscillations, von Neuman and Richtmeyer [22] introduced the concept of pseudoviscosity. The idea is to set an artificial dissipative pressure where the shock wave discontinuity is present. To do this, Neuman and Richtmeyer proposed the ansatz

$$Q = \begin{cases} (\alpha \Delta x)^2 \rho (\dot{\rho}/\rho)^2 & \dot{\rho} > 0 \\ 0 & \dot{\rho} < 0 \end{cases}$$

for non-relativistic one-dimensional hydrodynamics. Here, ρ is the mass density, Δx is the space grid size and α is a constant of the order of unity. In order to generalize the above pseudoviscosity for the relativistic SPH case, we replace the quantity $\dot{\rho}/\rho$ by $-\theta = -\partial_\mu u^\mu$ and Δx by h, where h is as before the width of the smoothing kernel W. More precisely, we
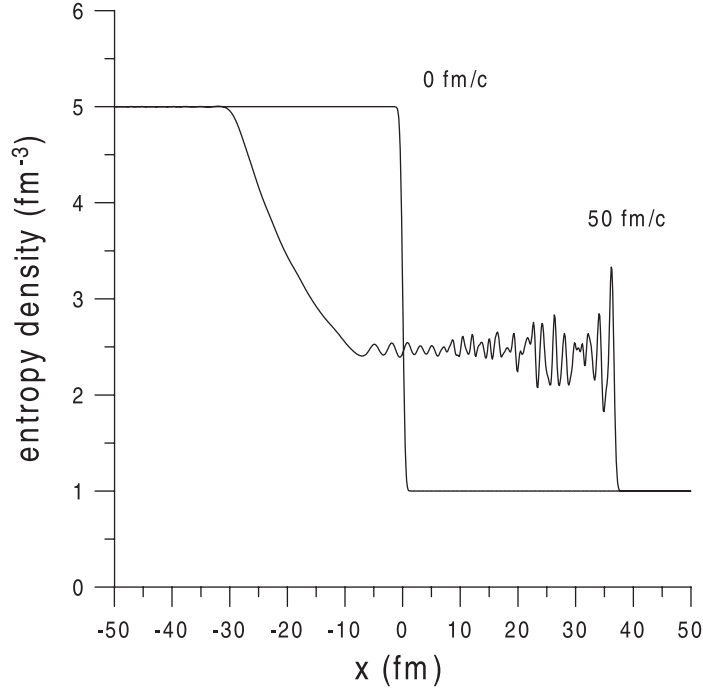
Figure 4. Shock wave formation in one-dimensional pion gas, calculated with SPH. No viscosity is used.

take the following form which is a slightly modified expression suggested in [13]:

$$Q = \begin{cases} P[-\alpha h\theta + \beta(h\theta)^2] & \theta < 0 \\ 0 & \theta \geqslant 0 \end{cases} \tag{57}$$

where P is the pressure. As mentioned before, Q is equivalent to the bulk viscosity and therefore there is no heat flow associated with it. What this artificial viscosity does is to convert the collective flow energy into the microscopic thermal energy. Consequently, the total energy, that is, the sum of the collective flow energy and the internal thermal energy is still conserved.

Figure 5 is the solution of the same problem as figure 4, but with entropy production taken into account. In this calculation, the pseudoviscosity parameters have been chosen as

$$\alpha = 2 \qquad \beta = 4$$

and $h = 0.5$ fm for 1000 SPH particles. As we see, the spurious oscillations have been damped out by the artificial viscosity, which also gives a finite width (of a few h) to the shock front.

It is known that the overall energy- and momentum-flux conservation relates the ratio s_2/s_1 of entropy densities after and before the shock to the velocity v_s of the shock front (Hugoniot–Rankine relation). For the pion gas equation of state (55) we have

$$\frac{s_2^*}{s_1^*} = \frac{2}{3^{3/4}} v_s \frac{\left(9v_s^2 - 1\right)^{1/4}}{\left(1 - v_s^2\right)^{5/4}}. \tag{58}$$

In figure 6, we show the velocity of the shock front obtained in our SPH calculations as a function of the entropy ratio (full circles). Each point corresponds to a different initial
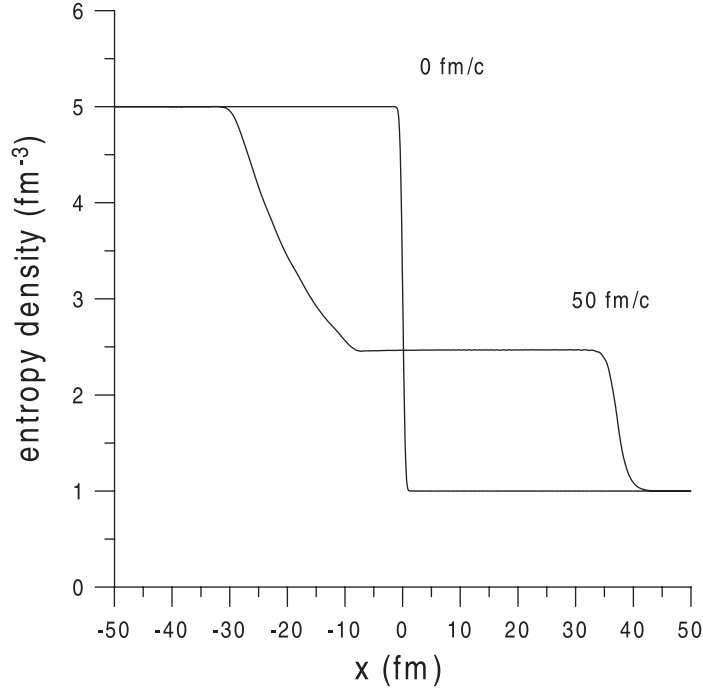
Figure 5. Same as in figure 4, after the introduction of the pseudoviscosity Q in the SPH calculation.

condition. They are compared with the Hugoniot–Rankine curve, equation (58). The accordance shows that our SPH calculation reproduces faithfully the conservation of kinetic energy and momentum of the flow through the shock front.

3.2.2. Rarefaction shock. When the fluid presents a first-order phase transition, a discontinuity appears in the expansion regime. This kind of shock wave has been discussed in connection with the QGP–hadron phase transition [15–17]. In the present example we assume that the hadron phase is a pion gas, and use a simple bag-model equation of state for the QGP,

$$P = \begin{cases} \frac{1}{30}\pi^2 T^4 & T \leqslant T_{\rm c} \\ \frac{1}{30}\pi^2 r T^4 - B & T \geqslant T_{\rm c} \end{cases}$$

where B is the bag constant, $r = \frac{37}{3}$ for two quark flavours, and $T_{\rm c}$ is the critical temperature. In the calculations below we use $B = 400$ MeV fm^{-3}.

The rarefaction shock occurs in the mixed-phase region (thermodynamically anomalous matter). In this case we need the artificial viscosity Q to be negative during expansion, and zero for compression. So, for $T = T_{\rm c}$ we use

$$Q = \begin{cases} 0 & \theta \leqslant 0 \\ P_{\rm c}\left[-\alpha h\theta - \beta(h\theta)^2\right] & \theta > 0 \end{cases}$$

where $P_{\rm c}$ is the critical pressure. In the single-phase regions (normal matter) the artificial viscosity is the same as before. In figure 7, we show the result of an SPH calculation and compare it with the analytic solution. The shock front is smoothed out by the pseudoviscosity,
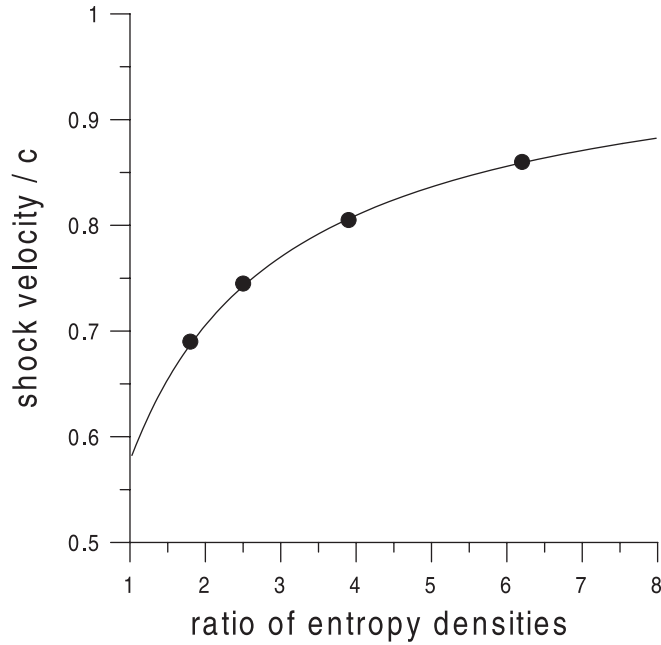
Figure 6. Test of Hugoniot–Rankine relation. The circles are results of SPH calculations with different initial conditions, and the full curve is equation (58).

but the overall features of the expansion process are well described. In this calculation we used 3600 particles and $h = 0.5$ fm. The pseudoviscosity parameters were $\alpha = \beta = 4$.

4. Discussion and perspectives

In the usual hydrodynamic computations using space grids, the symmetry of the problem is often a crucial factor in performing a calculation of reasonable size. The SPH method cures this aspect and furnishes a robust algorithm which is particularly appropriate to the description of processes where rapid expansions of the fluid should be treated. In this paper, we formulated an entropy-based SPH description of relativistic hydrodynamics. We have shown that this approach is very promising for the study of ultrarelativistic nucleus–nucleus collision processes. The equations of motion are derived by a variational procedure from the SPH model action with respect to the Lagrangian comoving coordinates. This guarantees that the method furnishes the maximal efficiency for a given number of degrees of freedom, keeping strict energy and momentum conservation. For this reason, solutions can be obtained with a very reasonable precision using a relatively small number of SPH particles. This is the basic advantage of the present method, when we analyse the event-by-event dynamics of relativistic heavy-ion collisions.

On the other hand, the precision of this method increases rather slowly with the number of SPH particles. Therefore, a relatively large number of particles is required if one wants a very precise numerical solution. However, for applications to RHIC physics we may need only a rather crude precision, especially if we consider the dubious validity of rigorous hydrodynamics. For a calculation with typically 10% errors, the SPH algorithm presented here furnishes a very efficient tool to study flow phenomena in RHIC physics.
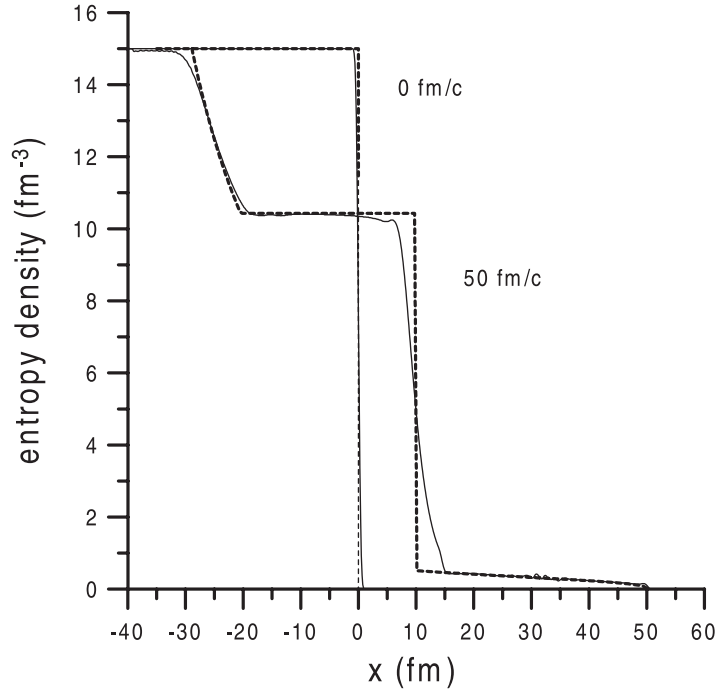
Figure 7. Comparison of the SPH results (full curves) with the analytical solution (broken curves) for the formation of a rarefaction shock.

A fundamental difficulty of relativistic hydrodynamics for a viscous fluid [23, 24] is that the dissipation term causes an intrinsic instability in the system described by equation (23). This instability basically comes from the fact that the dissipation term contains θ (see equations (24) and (33)), so that it necessarily introduces a third-order time derivative of position into the equations of motion. This means that we have to specify, at least, part of the acceleration as the initial condition. Even if we specify the initial acceleration, the requirement of the internal self-consistency among the equations above leads to intrinsically unstable solutions. Israel proposed [23, 24] to cure these difficulties by introducing higher-order thermodynamics with respect to deviations from the equilibrium. In the examples presented in the present paper, we did not address this question and simply estimated the quantity θ from the quantities one time step before. In practice, this causes no numerical instability and the behaviour of the solution is quite satisfactory.

For the future application of the present program, we need to specify more realistic initial conditions and also to relate the final state to the physical observable quantities, such as particle spectra. The first point is now being in progress by introducing the initial energy and momentum distributions of the matter using the NEXUS model [25]. As for the second point, that is, the problem of particle production, the possibility of incorporation of the continuous emission mechanism [26] is being studied.

Acknowledgments

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